3. Consider **i**, **j**, and **k** as the basis of the three dimensional vector space. Now construct a new basis **a**=**i**+**j**, **b**=**i**+**k**, **c**=**j**+**k**.

a. Construct the change of basis matrix.

Solution:

Let I={**i**, **j**, **k**} and A={**a**, **b**, **c**}

$$\vec{a} = I\updownarrow\begin{pmatrix}1\\1\\0\end{pmatrix} \qquad \vec{b} = I\updownarrow\begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \vec{c} = I\updownarrow\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$\therefore$ The change of basis matrix is

$$U = I\updownarrow\overset{\overleftrightarrow{A}}{\begin{pmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix}}$$

To calculate inverse of U :

$$\left(\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\1 & 0 & 1 & 0 & 1 & 0\\0 & 1 & 1 & 0 & 0 & 1\end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\1 & -1 & 0 & 0 & 1 & -1\\0 & 1 & 1 & 0 & 0 & 1\end{array}\right) \quad \text{(2nd row - 3rd row} \rightarrow \text{2nd row)}$$

$$\rightarrow \left(\begin{array}{ccc|ccc}2 & 0 & 0 & 1 & 1 & -1\\-2 & 2 & 0 & 0 & -2 & 2\\0 & 1 & 1 & 0 & 0 & 1\end{array}\right) \quad \text{(2nd row} \times -2 \rightarrow \text{2nd row)}$$

$$\rightarrow \left(\begin{array}{ccc|ccc}2 & 0 & 0 & 1 & 1 & -1\\0 & 2 & 0 & 1 & -1 & 1\\0 & 1 & 1 & 0 & 0 & 1\end{array}\right) \quad \text{(1st row + 2nd row} \rightarrow \text{2nd row)}$$

$$\rightarrow \left(\begin{array}{ccc|ccc}2 & 0 & 0 & 1 & 1 & -1\\0 & 2 & 0 & 1 & -1 & 1\\0 & 2 & 2 & 0 & 0 & 2\end{array}\right) \quad \text{(3rd row} \times 2 \rightarrow \text{3rd row)}$$

$$\rightarrow \left(\begin{array}{ccc|ccc}2 & 0 & 0 & 1 & 1 & -1\\0 & 2 & 0 & 1 & -1 & 1\\0 & 0 & 2 & -1 & 1 & 1\end{array}\right) \quad \text{(3rd row} - \text{2nd row} \rightarrow \text{3rd row)}$$

$$\rightarrow \left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 1 & -1\\0 & 1 & 0 & 1 & -1 & 1\\0 & 0 & 1 & -1 & 1 & 1\end{array}\right) \times \frac{1}{2} \quad \text{(whole matrix} \times \frac{1}{2}\text{)}$$

$$\therefore \quad U^{-1} = A \updownarrow \overset{\overset{I}{\longleftrightarrow}}{\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}}$$

b. Write vector **r** = **i**+2**j**+3**k** in terms of **a**, **b**, and **c**.
Solution:

$$\vec{r} = \vec{i} + 2\vec{j} + 3\vec{k} = I \updownarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$\therefore$ In A - representation :

$$\vec{r} = A \updownarrow \overset{\overset{I}{\longleftrightarrow}}{\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}} \quad I \updownarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$I$$

$$= A \updownarrow \frac{1}{2}\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$$

$$= A \updownarrow \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$= \underline{\underline{0\vec{a} + 1\vec{b} + 2\vec{c}}}$$

c. Convert the following matrix from the old ijk-representation to the new abc-representation:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Solution:

In I - representation :

$$I\updownarrow\overset{\overset{I}{\leftrightarrow}}{M} \;=\; I\updownarrow\overset{\overset{I}{\longleftrightarrow}}{\begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}}$$

$$\therefore\;\; A\updownarrow\overset{\overset{A}{\leftrightarrow}}{M} \;=\; A\updownarrow\overset{\overset{I}{\longleftrightarrow}}{\frac{1}{2}\begin{pmatrix}1&1&-1\\1&-1&1\\-1&1&1\end{pmatrix}}\; I\updownarrow\overset{\overset{I}{\longleftrightarrow}}{\begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}}\; I\updownarrow\overset{\overset{A}{\longleftrightarrow}}{\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}}$$

$$= A\updownarrow\overset{\overset{I}{\longleftrightarrow}}{\frac{1}{2}\begin{pmatrix}1&1&-1\\1&-1&1\\-1&1&1\end{pmatrix}}\; I\updownarrow\overset{\overset{A}{\longleftrightarrow}}{\begin{pmatrix}1&2&1\\1&0&1\\1&2&1\end{pmatrix}}$$

$$= A\updownarrow\overset{\overset{A}{\longleftrightarrow}}{\frac{1}{2}\begin{pmatrix}1&0&1\\1&4&1\\1&0&1\end{pmatrix}}$$

$$= A\updownarrow\underline{\underline{\overset{\overset{A}{\longleftrightarrow}}{\begin{pmatrix}\frac{1}{2}&0&\frac{1}{2}\\\frac{1}{2}&2&\frac{1}{2}\\\frac{1}{2}&0&\frac{1}{2}\end{pmatrix}}}}$$