Gasiorowicz 3rd edition April 23, 2008
P10-11.

Consider two spin ½ particles, whose spins are described by the Pauli operators $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. Let **e** be the unit vector connecting the two particles and define the operator

$$S_{12} = 3(\sigma_1 \cdot \hat{e})(\sigma_2 \cdot \hat{e}) - \sigma_1 \cdot \sigma_2$$

Show that if the two particles are in an S=0 state (singlet) then

$S_{12}X_{singlet}=0$

Show that for a triplet state

$(S_{12}-2)(S_{12}+4)\ X_{singlet} = 0$

(Hint: Choose e along the z-axis)

Solution:

We will solve the problem in $S_{1z} - S_{2z}$ (non - interacting) representation.

Choose $\hat{e}$ along the z - axis.

$$\sigma_1 \cdot \hat{e} = \sigma_{1z} \text{ and } \sigma_2 \cdot \hat{e} = \sigma_{2z}$$

$$\begin{aligned}
S_{12} &= 3(\sigma_1 \cdot \hat{e})(\sigma_2 \cdot \hat{e}) - \sigma_1 \cdot \sigma_2 \\
&= 3\sigma_{1z}\sigma_{2z} - (\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z}) \\
&= 2\sigma_{1z}\sigma_{2z} - \sigma_{1x} \cdot \sigma_{2x} - \sigma_{1y} \cdot \sigma_{2y}) \\
&= 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_2 - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_2 - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_1 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_2
\end{aligned}$$

Note that you cannot multiply the two matrices together, because they belong to two different particls.

In $S_{1z} - S_{2z}$ (non - interacting) representation,

$$X_{singlet} = \frac{1}{\sqrt{2}}(|\uparrow>|\downarrow> - |\downarrow>|\uparrow>) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2\right]$$

$$S_{12}X_{singlet} = \left[2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_2 - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_2 - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_1 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_2\right] \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2\right]$$

$$\therefore \; S_{12} X_{singlet}$$

$$= \frac{1}{\sqrt{2}}\left[2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_2 \begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_2 \begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}_2 \begin{pmatrix}0\\1\end{pmatrix}_2 \right]$$

$$- \frac{1}{\sqrt{2}}\left[2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_2 \begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_2 \begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}_2 \begin{pmatrix}1\\0\end{pmatrix}_2 \right]$$

$$= \frac{1}{\sqrt{2}}\left[2\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\-1\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}0\\i\end{pmatrix}_1 \begin{pmatrix}-i\\0\end{pmatrix}_2 \right] - \frac{1}{\sqrt{2}}\left[2\begin{pmatrix}0\\-1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}-i\\0\end{pmatrix}_1 \begin{pmatrix}0\\i\end{pmatrix}_2 \right]$$

$$= \frac{1}{\sqrt{2}}\left[2\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\-1\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 - i\cdot i\begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}-1\\0\end{pmatrix}_2 \right] - \frac{1}{\sqrt{2}}\left[2\begin{pmatrix}0\\-1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 - i\cdot i\begin{pmatrix}-1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 \right]$$

$$= \frac{1}{\sqrt{2}}\left[2\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\-1\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}-1\\0\end{pmatrix}_2 \right] - \frac{1}{\sqrt{2}}\left[2\begin{pmatrix}0\\-1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}-1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 \right]$$

$$= \frac{1}{\sqrt{2}}\left[-2\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 \right] - \frac{1}{\sqrt{2}}\left[-2\begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 \right]$$

$$= \frac{1}{\sqrt{2}}\left[-2\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 + 2\begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 \right]$$

$$= \underline{\underline{0}}$$

We can use the above "mechanical method" to prove $(S_{12}-2)(S_{12}+4)\,X_{singlet} = 0$. Note that there are three triplet states:

$$X_{triplet\,1} = \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2$$

$$X_{triplet\,2} = \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2$$

$$X_{triplet\,3} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 \right]$$

And you have to prove each case independently. Since we have demonstrated the mechanical method already, we will prove this by a more efficient approach:

In $S_{1z} - S_{2z}$ (non - interacting) representation.

We first note that

$$(\vec{S}_1+\vec{S}_2)^2 = \frac{\hbar^2}{4}(\vec{\sigma}_1+\vec{\sigma}_2)^2 = \frac{\hbar^2}{4}(\vec{\sigma}_1{}^2+\vec{\sigma}_2{}^2+2\vec{\sigma}_1\cdot\vec{\sigma}_2)$$

$$\vec{\sigma}^2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\therefore (\vec{S}_1+\vec{S}_2)^2 = \frac{\hbar^2}{4}(\vec{\sigma}_1{}^2+\vec{\sigma}_2{}^2+2\vec{\sigma}_1\cdot\vec{\sigma}_2) = \frac{\hbar^2}{4}(6I+2\vec{\sigma}_1\cdot\vec{\sigma}_2)$$

$$2\vec{\sigma}_1\cdot\vec{\sigma}_2 = \frac{4}{\hbar^2}(\vec{S}_1+\vec{S}_2)^2 - 6I$$

$$\Rightarrow \vec{\sigma}_1\cdot\vec{\sigma}_2 = \frac{2}{\hbar^2}(\vec{S}_1+\vec{S}_2)^2 - 3I$$

$$S_{12} = 3(\sigma_1\cdot\hat{e})(\sigma_2\cdot\hat{e}) - \sigma_1\cdot\sigma_2 = 3\sigma_{1z}\sigma_{2z} - \vec{\sigma}_1\cdot\vec{\sigma}_2$$

$$= 3\sigma_{1z}\sigma_{2z} - \left(\frac{2}{\hbar^2}(\vec{S}_1+\vec{S}_2)^2 - 3I\right)$$

For triplet state, $S=1$. $\therefore$ $(\vec{S}_1+\vec{S}_2)^2 = 1(1+1)\hbar^2 = 2\hbar^2$

$$S_{12} = 3\sigma_{1z}\sigma_{2z} - \left(\frac{2}{\hbar^2}(\vec{S}_1+\vec{S}_2)^2 - 3I\right) = 3\sigma_{1z}\sigma_{2z} - \left(\frac{2}{\hbar^2}\cdot 2\hbar^2 - 3I\right)$$

$$= 3\sigma_{1z}\sigma_{2z} - I$$

Case1.

$$X_{\text{triplet}\,1} = \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2$$

$$S_{12}X_{\text{triplet}\,1} = (3\sigma_{1z}\sigma_{2z} - I)\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 = (3-1)\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 = 2X_{\text{triplet}\,1}$$

$$\therefore\ (S_{12}-2)X_{\text{triplet}\,1} = 0$$

Case 2.

$$X_{\text{triplet}\,2} = \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2$$

$$S_{12}X_{\text{triplet}\,2} = (3\sigma_{1z}\sigma_{2z} - I)\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 = [3(-1\cdot -1)-1]\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 = 2X_{\text{triplet}\,2}$$

$$\therefore\ (S_{12}-2)X_{\text{triplet}\,2} = 0$$

Case1.

$$X_{\text{triplet}\,1} = \begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2$$

$$S_{12}X_{\text{triplet}\,1} = (3\sigma_{1z}\sigma_{2z} - I)\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 = (3-1)\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 = 2X_{\text{triplet}\,1}$$

$$\therefore\ (S_{12} - 2)X_{\text{triplet}\,1} = 0$$

Case 2.

$$X_{\text{triplet}\,2} = \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2$$

$$S_{12}X_{\text{triplet}\,2} = (3\sigma_{1z}\sigma_{2z} - I)\begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 = [3(-1\cdot -1) - 1]\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 = 2X_{\text{triplet}\,2}$$

$$\therefore\ (S_{12} - 2)X_{\text{triplet}\,2} = 0$$

Case 3.

$$\begin{aligned}
X_{\text{triplet}\,3} &= (3\sigma_{1z}\sigma_{2z} - I)\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2\right] \\
&= [3(1\cdot -1) - 1]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 + [3(-1\cdot 1) - 1]\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 \\
&= -4\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 + (-4)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2 \\
&= -4\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1 \begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_1 \begin{pmatrix}1\\0\end{pmatrix}_2\right] \\
&= -4X_{\text{triplet}\,3}
\end{aligned}$$

$$\therefore\ (S_{12} + 4)X_{\text{triplet}\,2} = 0$$

To include all three cases, we have

$$(S_{12} - 2)\,(S_{12} + 4)X_{\text{triplet}\,2} = 0$$