Gasiorowicz 3rd edition April 23, 2008
P10-12.

In a low energy neutron-proton system (which has zero orbital angular momentum) the potential energy is given by

$$V(r) = V_1(r) + V_2(r)\left(3\frac{(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})}{r^2} - \vec{\sigma}_1\cdot\vec{\sigma}_2\right) + V_3(r)\vec{\sigma}_1\cdot\vec{\sigma}_2$$

Where r is the vector connecting the two particles. Calculate the potential energy for the neutron-proton system (a) In the spin singlet state, and (b) In the triplet state.

Solution: In $S_{1z} - S_{2z}$ (non-interacting) representation.

$$(\vec{S}_1 + \vec{S}_2)^2 = \frac{\hbar^2}{4}(\vec{\sigma}_1 + \vec{\sigma}_2)^2 = \frac{\hbar^2}{4}(\vec{\sigma}_1{}^2 + \vec{\sigma}_2{}^2 + 2\vec{\sigma}_1\cdot\vec{\sigma}_2)$$

$$\vec{\sigma}^2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\therefore (\vec{S}_1 + \vec{S}_2)^2 = \frac{\hbar^2}{4}(\vec{\sigma}_1{}^2 + \vec{\sigma}_2{}^2 + 2\vec{\sigma}_1\cdot\vec{\sigma}_2) = \frac{\hbar^2}{4}(6I + 2\vec{\sigma}_1\cdot\vec{\sigma}_2)$$

$$2\vec{\sigma}_1\cdot\vec{\sigma}_2 = \frac{4}{\hbar^2}(\vec{S}_1 + \vec{S}_2)^2 - 6I$$

$$\Rightarrow \vec{\sigma}_1\cdot\vec{\sigma}_2 = \frac{2}{\hbar^2}(\vec{S}_1 + \vec{S}_2)^2 - 3I$$

Choosing $\vec{r}$ as the z-axis,

$$V(r) = V_1(r) + V_2(r)\left(3\frac{(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})}{r^2} - \vec{\sigma}_1\cdot\vec{\sigma}_2\right) + V_3(r)\vec{\sigma}_1\cdot\vec{\sigma}_2$$

$$= V_1(r) + V_2(r)\left(3\frac{(\vec{\sigma}_{1z}\vec{\sigma}_{2z}r^2)}{r^2} - \vec{\sigma}_1\cdot\vec{\sigma}_2\right) + V_3(r)\vec{\sigma}_1\cdot\vec{\sigma}_2$$

$$= V_1(r) + 3V_2(r)\vec{\sigma}_{1z}\vec{\sigma}_{2z} + [V_3(r) - V_2(r)]\vec{\sigma}_1\cdot\vec{\sigma}_2$$

$$= V_1(r) + 3V_2(r)\vec{\sigma}_{1z}\vec{\sigma}_{2z} + [V_3(r) - V_2(r)]\left(\frac{2}{\hbar^2}(\vec{S}_1 + \vec{S}_2)^2 - 3I\right)$$

(a) In singlet state,

$$X_{singlet} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]$$

$$\vec{\sigma}_{1z}\vec{\sigma}_{2z}X_{singlet} = \vec{\sigma}_{1z}\vec{\sigma}_{2z}\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]$$

$$= [(1\cdot -1)]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 - [(-1\cdot 1)]\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2$$

$$= -1\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 - (-1)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2$$

$$= -1 \cdot \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 - \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right]$$

$$= -X_{\sin glet}$$

$S = 0$ for singlet, $\therefore (\vec{S}_1 + \vec{S}_2)^2 = 0(0+1)\hbar^2 = 0$

$$\therefore V(r) = V_1(r) + 3V_2(r)\vec{\sigma}_{1z}\vec{\sigma}_{2z} + [V_3(r) - V_2(r)]\left(\frac{2}{\hbar^2}(\vec{S}_1 + \vec{S}_2)^2 - 3I\right)$$

$$= V_1(r) + 3V_2(r)(-1) + [V_3(r) - V_2(r)](0 - 3I)$$

$$= V_1(r) - 3V_2(r) - [V_3(r) - V_2(r)](3)$$

$$= \underline{\underline{V_1(r) - 3V_3(r)}}$$

(b) In triplet state,

$S = 0$ for singlet, $\therefore (\vec{S}_1 + \vec{S}_2)^2 = 1(1+1)\hbar^2 = 2\hbar^2$

$$\therefore V(r) = V_1(r) + 3V_2(r)\vec{\sigma}_{1z}\vec{\sigma}_{2z} + [V_3(r) - V_2(r)]\left(\frac{2}{\hbar^2}\cdot 2\hbar^2 - 3I\right)$$

$$= V_1(r) + 3V_2(r)\vec{\sigma}_{1z}\vec{\sigma}_{2z} + [V_3(r) - V_2(r)](4 - 3)$$

$$= V_1(r) + V_2(r)[3\vec{\sigma}_{1z}\vec{\sigma}_{2z} - 1] + V_3(r)$$

Case1.

$$X_{\text{triplet 1}} = \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2$$

$$3\sigma_{1z}\sigma_{2z}X_{\text{triplet 1}} = (3\sigma_{1z}\sigma_{2z})\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 = (3)\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 = 3X_{\text{triplet 1}}$$

$$V(r) = V_1(r) + V_2(r)[3-1] + V_3(r) = \underline{\underline{V_1(r) + 2V_2(r) + V_3(r)}}$$

Case 2.

$$X_{\text{triplet 2}} = \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2$$

$$3\sigma_{1z}\sigma_{2z}X_{\text{triplet 2}} = (3\sigma_{1z}\sigma_{2z})\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 = [3(-1\cdot -1)]\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 = 3X_{\text{triplet 2}}$$

$$V(r) = V_1(r) + V_2(r)[3-1] + V_3(r) = \underline{\underline{V_1(r) + 2V_2(r) + V_3(r)}}$$

Case 3.

$$X_{\text{triplet 3}} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]$$

$$3\sigma_{1z}\sigma_{2z}X_{\text{triplet 3}} = (3\sigma_{1z}\sigma_{2z})\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]$$

$$= [3(1\cdot -1)]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + [3(-1\cdot 1)]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2$$

$$= -3\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + (-3)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2$$

$$= -3\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2\right]$$

$$= -3X_{\text{triplet 3}}$$

$$V(r) = V_1(r) + V_2(r)[-3-1] + V_3(r) = \underline{\underline{V_1(r) - 4V_2(r) + V_3(r)}}$$