Gasiorowicz 3rd edition April 24, 2008

P10-14.

A particle of spin 1 moves in a central potential of the form

$$V(r) = V_1(r) + \frac{\vec{S}\cdot\vec{L}}{\hbar^2} V_2(r) + \frac{(\vec{S}\cdot\vec{L})^2}{\hbar^4} V_3(r)$$

What are the values of V(r) in the states J=L+1, L, and L+1?

Solution:

$$V(r) = V_1(r) + \frac{\vec{S}\cdot\vec{L}}{\hbar^2} V_2(r) + \frac{(\vec{S}\cdot\vec{L})^2}{\hbar^4} V_3(r)$$

$$\vec{S}\cdot\vec{L} = \frac{1}{2}[(\vec{S}+\vec{L})^2 - \vec{S}^2 - \vec{L}^2] = \frac{1}{2}[\vec{J}^2 - \vec{S}^2 - \vec{L}^2]$$

$$\text{With spin} = 1,\ \vec{S}^2 = 1(1+1)\hbar^2 = 2\hbar^2$$

$$\vec{S}\cdot\vec{L} = \frac{1}{2}[\vec{J}^2 - \vec{L}^2 - 2\hbar^2]$$

$$\text{Case 1. } J = L+1$$

$$\vec{J}^2 = J(J+1)\hbar^2 = (L+1)(L+2)\hbar^2$$

$$\vec{L}^2 = L(L+1)\hbar^2$$

$$\begin{aligned}\vec{S}\cdot\vec{L} = \frac{1}{2}[\vec{J}^2 - \vec{L}^2 - 2\hbar^2] &= \frac{1}{2}[(L+1)(L+2)\hbar^2 - (L)(L+1)\hbar^2 - 2\hbar^2] \\ &= \frac{1}{2}[(L+1)(L+2-L)\hbar^2 - 2\hbar^2] \\ &= \frac{1}{2}[(2L+2)\hbar^2 - 2\hbar^2] \\ &= L\hbar^2\end{aligned}$$

$$V(r) = V_1(r) + \frac{L\hbar^2}{\hbar^2} V_2(r) + \frac{(L\hbar^2)^2}{\hbar^4} V_3(r) = \underline{\underline{V_1(r) + LV_2(r) + L^2 V_3(r)}}$$

$$\text{Case 1. } J = L$$

$$\vec{J}^2 = J(J+1)\hbar^2 = L(L+1)\hbar^2$$

$$\vec{L}^2 = L(L+1)\hbar^2$$

$$\begin{aligned}\vec{S}\cdot\vec{L} = \frac{1}{2}[\vec{J}^2 - \vec{L}^2 - 2\hbar^2] &= \frac{1}{2}[(L)(L+1)\hbar^2 - (L)(L+1)\hbar^2 - 2\hbar^2] \\ &= -\hbar^2\end{aligned}$$

$$V(r) = V_1(r) + \frac{-\hbar^2}{\hbar^2} V_2(r) + \frac{(-\hbar^2)^2}{\hbar^4} V_3(r) = \underline{\underline{V_1(r) - V_2(r) + V_3(r)}}$$

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Case 3. $J = L - 1$

$$\vec{J}^2 = J(J+1)\hbar^2 = (L-1)L\hbar^2$$

$$\vec{L}^2 = L(L+1)\hbar^2$$

$$\begin{aligned}
\vec{S}\cdot\vec{L} = \frac{1}{2}[\vec{J}^2 - \vec{L}^2 - 2\hbar^2] &= \frac{1}{2}[(L-1)L\hbar^2 - (L)(L+1)\hbar^2 - 2\hbar^2] \\
&= \frac{1}{2}[L(L-1-L-1)\hbar^2 - 2\hbar^2] \\
&= \frac{1}{2}(-2L\hbar^2 - 2\hbar^2) \\
&= -(L\hbar^2 + \hbar^2)
\end{aligned}$$

$$\begin{aligned}
V(r) &= V_1(r) + \frac{-(L\hbar^2 + \hbar^2)}{\hbar^2}V_2(r) + \frac{\left[-(L\hbar^2 + \hbar^2)\right]^2}{\hbar^4}V_3(r) \\
&= \underline{\underline{V_1(r) - (L+1)V_2(r) + (L+1)^2 V_3(r)}}
\end{aligned}$$