Gasiorowicz 3rd edition April 24, 2008

P10-2.

Find the normalized eigenspinors of the 2×2 matrix

$$\begin{pmatrix} \cos\alpha & \sin\alpha e^{-i\beta} \\ \sin\alpha e^{i\beta} & -\cos\alpha \end{pmatrix}$$

Solution:

Let the eigenvalue be λ

$$\det\begin{pmatrix} \cos\alpha-\lambda & \sin\alpha e^{-i\beta} \\ \sin\alpha e^{i\beta} & -\cos\alpha-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (\cos\alpha-\lambda)(-\cos\alpha-\lambda)-(\sin\alpha e^{-i\beta})(\sin\alpha e^{i\beta}) = 0$$

$$\Rightarrow -(\cos\alpha-\lambda)(\cos\alpha+\lambda)-\sin^2\alpha = 0$$

$$\Rightarrow -(\cos^2\alpha-\lambda^2)-\sin^2\alpha = 0$$

$$\Rightarrow \lambda^2-(\cos^2\alpha+\sin^2\alpha) = 0$$

$$\Rightarrow \lambda^2-1 = 0$$

$$\Rightarrow \lambda = \pm 1$$

Let the eigenvector be $\begin{pmatrix} u \\ v \end{pmatrix}$

Case 1: $\lambda = 1$

$$\begin{pmatrix} \cos\alpha & \sin\alpha e^{-i\beta} \\ \sin\alpha e^{i\beta} & -\cos\alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{cases} u\cos\alpha + v\sin\alpha e^{-i\beta} = u \\ u\sin\alpha e^{i\beta} - v\cos\alpha = v \end{cases}$$

$$\Rightarrow \begin{cases} u\cos^2\alpha + v\sin\alpha\cos\alpha e^{-i\beta} = u\cos\alpha \\ u\sin^2\alpha e^{i\beta} - v\sin\alpha\cos\alpha = v\sin\alpha \end{cases}$$

$$\Rightarrow \begin{cases} ue^{i\beta}\cos^2\alpha + v\sin\alpha\cos\alpha = ue^{i\beta}\cos\alpha \\ u\sin^2\alpha e^{i\beta} - v\sin\alpha\cos\alpha = v\sin\alpha \end{cases}$$

$$\Rightarrow ue^{i\beta} = ue^{i\beta}\cos\alpha + v\sin\alpha\cos\alpha$$

$$\Rightarrow u(1-\cos\alpha)e^{i\beta} = v\sin\alpha\cos\alpha$$

$$\Rightarrow u\cdot 2\sin^2\frac{\alpha}{2}e^{i\beta} = v\cdot 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$\Rightarrow u\cdot\sin\frac{\alpha}{2}e^{i\beta} = v\cdot\cos\frac{\alpha}{2}$$

Choose $u = 1, v = e^{i\beta}\tan\frac{\alpha}{2}$. $\therefore$ The eigenvector is $\begin{pmatrix} 1 \\ e^{i\beta}\tan\frac{\alpha}{2} \end{pmatrix}$

Magnitude of this vector $=\sqrt{|1|^2+|e^{i\beta}\tan\frac{\alpha}{2}|^2} = \sqrt{1+\tan^2\frac{\alpha}{2}} = \frac{1}{\cos\frac{\alpha}{2}}$

Normalizing the vector, it becomes

$$\cos\frac{\alpha}{2}\begin{pmatrix} 1 \\ e^{i\beta}\tan\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix}$$

Case 2 : $\lambda = -1$

$$\begin{pmatrix} \cos\alpha & \sin\alpha e^{-i\beta} \\ \sin\alpha e^{i\beta} & -\cos\alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{cases} u\cos\alpha + v\sin\alpha e^{-i\beta} = -u \\ u\sin\alpha e^{i\beta} - v\cos\alpha = -v \end{cases}$$

$$\Rightarrow \begin{cases} u\cos^2\alpha + v\sin\alpha\cos\alpha e^{-i\beta} = -u\cos\alpha \\ u\sin^2\alpha e^{i\beta} - v\sin\alpha\cos\alpha = -v\sin\alpha \end{cases}$$

$$\Rightarrow \begin{cases} ue^{i\beta}\cos^2\alpha + v\sin\alpha\cos\alpha = -ue^{i\beta}\cos\alpha \\ u\sin^2\alpha e^{i\beta} - v\sin\alpha\cos\alpha = -v\sin\alpha \end{cases}$$

$$\Rightarrow ue^{i\beta} = -ue^{i\beta}\cos\alpha - v\sin\alpha\cos\alpha$$

$$\Rightarrow u(1+\cos\alpha)e^{i\beta} = -v\sin\alpha\cos\alpha$$

$$\Rightarrow u\cdot 2\cos^2\frac{\alpha}{2}e^{i\beta} = -v\cdot 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$\Rightarrow u\cdot\cos\frac{\alpha}{2}e^{i\beta} = -v\cdot\sin\frac{\alpha}{2}$$

Choose $u=1, v=-e^{i\beta}\cot\frac{\alpha}{2}$. $\therefore$ The eigenvector is $\begin{pmatrix} 1 \\ -e^{i\beta}\cot\frac{\alpha}{2} \end{pmatrix}$

Magnitude of this vector $=\sqrt{|1|^2+|-e^{i\beta}\cot\frac{\alpha}{2}|^2} = \sqrt{1+\cot^2\frac{\alpha}{2}} = \frac{1}{\sin\frac{\alpha}{2}}$

Normalizing the vector, it becomes

$$\sin\frac{\alpha}{2}\begin{pmatrix} 1 \\ -e^{i\beta}\cot\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} \sin\frac{\alpha}{2} \\ -e^{i\beta}\cos\frac{\alpha}{2} \end{pmatrix}$$

$\therefore$ The two eigenvectors are $\begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\beta}\sin\frac{\alpha}{2} \end{pmatrix}$ and $\begin{pmatrix} \sin\frac{\alpha}{2} \\ -e^{i\beta}\cos\frac{\alpha}{2} \end{pmatrix}$.