Gasiorowicz 3[rd] edition April 24, 2008
P10-4

Use material from Chapter 9 to construct the matrix representation of spin 3/2. Find the eigenstates in the representation in which S_z is diagonal.

Solution:

For spin 3/2, 2s+1=4. So we expect all matrices to be 4×4:

In S_z representation,

$$\vec{S}^2 = s(s+1)\hbar^2 = \frac{3}{2}\cdot\frac{5}{2}\hbar^2 = \frac{15}{4}\hbar^2$$

$$\therefore \vec{S}^2 = \begin{pmatrix} \frac{15}{4} & 0 & 0 & 0 \\ 0 & \frac{15}{4} & 0 & 0 \\ 0 & 0 & \frac{15}{4} & 0 \\ 0 & 0 & 0 & \frac{15}{4} \end{pmatrix}^{2}\hbar$$

$$S_z = m\hbar = \frac{3}{2}\hbar, \frac{1}{2}\hbar, -\frac{1}{2}\hbar, -\frac{3}{2}\hbar$$

$$\therefore S_z = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}\hbar$$

To determine S_x and S_y, we need to first find out $S_\pm$.

In S_z representation,

$$S_+ = \begin{pmatrix} 0 & \sqrt{\frac{3}{2}\left(\frac{5}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}\left(\frac{5}{2}\right)-\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{2}\left(\frac{5}{2}\right)-\left(-\frac{3}{2}\right)\left(-\frac{3}{2}+1\right)} \\ 0 & 0 & 0 & 0 \end{pmatrix}\hbar$$

$$= \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}\hbar$$

$$S_- = S_+{}^+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}^+ \hbar = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\hbar$$

$$\therefore S_x = \frac{1}{2}(S_+ + S_-) = \frac{1}{2}\left[\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}\hbar + \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\hbar\right]$$

$$= \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\hbar$$

$$= \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}\hbar$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{1}{2i}\left[\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}\hbar - \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\hbar\right]$$

$$= -\frac{i}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}\hbar$$

$$= \underline{\underline{\begin{pmatrix} 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix} i\hbar}}$$

The eigenstates in S_z representation are simply

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ (eigenvalue $\frac{3}{2}$), $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ (eigenvalue $\frac{1}{2}$) , $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ (eigenvalue $-\frac{1}{2}$), and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ (eigenvalue $-\frac{3}{2}$)