Gasiorowicz 3rd edition April 24, 2008
P10-5

Consider the spinor $\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}$. What is the probability that a measurement of $(3S_x+4S_y)/5$ yields the value $-\hbar/2$?

Solution:
According to the wordings of the problem, $(3S_x+4S_y)/5$ should have one of its eigenvalue equals to $-\hbar/2$. However, the given spinor is in S_z representation. Hence, to solve this problem, we just need to expand the given spinor in terms of the eigenvectors of $(3S_x+4S_y)/5$.

In S_z representation,

$$S_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\frac{\hbar}{2} \text{ and } S_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\frac{\hbar}{2}$$

$$\therefore \frac{3S_x+4S_y}{5} = \frac{1}{5}\left[3\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\frac{\hbar}{2} + 4\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\frac{\hbar}{2}\right]$$

$$= \frac{\hbar}{10}\begin{pmatrix}0 & 3-4i\\3+4i & 0\end{pmatrix}$$

Let λ be the eigenvalues,

$$\det\begin{pmatrix}-\lambda & \frac{\hbar}{10}(3-4i)\\ \frac{\hbar}{10}(3+4i) & -\lambda\end{pmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - \frac{\hbar}{10}(3+4i)\cdot\frac{\hbar}{10}(3-4i) = 0$$

$$\Rightarrow \lambda^2 - \left(\frac{\hbar}{10}\right)^2 \cdot 25 = 0$$

$$\Rightarrow \lambda^2 = \frac{\hbar^2}{4}$$

$$\Rightarrow \lambda = \pm\frac{\hbar}{2} \quad \text{as expected.}$$

Now figure out the eigenvectors in S_z representation.

$$\frac{\hbar}{10}\begin{pmatrix}0 & 3-4i\\3+4i & 0\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}u\\v\end{pmatrix} \quad \Rightarrow \quad \begin{cases}\frac{1}{5}(3-4i)v = u\\ \frac{1}{5}(3+4i)u = v\end{cases}$$

Let $u = 1$, $v = \frac{1}{5}(3+4i)$

$\therefore \chi_+ = \begin{pmatrix}1\\ \frac{1}{5}(3+4i)\end{pmatrix}$ and it has a magnitude of $\sqrt{2}$. Normalizing it, we have

$$\chi_+ = \frac{1}{5\sqrt{2}}\begin{pmatrix}5\\3+4i\end{pmatrix}$$

$$\frac{\hbar}{10}\begin{pmatrix} 0 & 3-4i \\ 3+4i & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{cases} \frac{1}{5}(3-4i)v = -u \\ \frac{1}{5}(3+4i)u = -v \end{cases}$$

Let $v = 1, u = -\frac{1}{5}(3+4i)$

$\therefore \chi_{-} = \begin{pmatrix} -\frac{1}{5}(3+4i) \\ 1 \end{pmatrix}$ and it has a magnitude of $\sqrt{2}$. Normalizing it, we have

$$\chi_{-} = \frac{1}{5\sqrt{2}}\begin{pmatrix} -3-4i \\ 5 \end{pmatrix}$$

Therefore we can construct the transformation matrix (using our "representation tagger"):

$$U = S_z \updownarrow \overset{\longleftarrow \chi \longrightarrow}{\frac{1}{5\sqrt{2}}\begin{pmatrix} 5 & -3-4i \\ 3+4i & 5 \end{pmatrix}}$$

U should be unitary:

$$U^{-1} = U^{+} = \chi \updownarrow \overset{\longleftarrow S_z \longrightarrow}{\frac{1}{5\sqrt{2}}\begin{pmatrix} 5 & 3-4i \\ -3+4i & 5 \end{pmatrix}}$$

We need to transform the given state vector ψ from S_z representation to χ representation:

$$\chi \updownarrow \psi = \chi \updownarrow \overset{\longleftarrow S_z \longrightarrow}{\frac{1}{5\sqrt{2}}\begin{pmatrix} 5 & 3-4i \\ -3+4i & 5 \end{pmatrix}} \quad S_z \updownarrow \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \chi \updownarrow \overset{\longleftarrow S_z \longrightarrow}{\frac{1}{5\sqrt{10}}\begin{pmatrix} 10+3-4i \\ -6+8i+5 \end{pmatrix}}$$

$$= \chi \updownarrow \overset{\longleftarrow S_z \longrightarrow}{\frac{1}{5\sqrt{10}}\begin{pmatrix} 13-4i \\ -1+8i \end{pmatrix}}$$

$\therefore$ Probability that a measurement of $\frac{3S_x + 4S_y}{5}$ yields the value $-\frac{\hbar}{2}$

$$= \left|\frac{-1+8i}{5\sqrt{10}}\right|^2$$

$$= \frac{1+64}{250}$$

$$= \frac{65}{250}$$

$$= \underline{\underline{\frac{13}{50}}}$$