Gasiorowicz 3rd edition April 24, 2008
P10-6

Consider a spin ½ system represented by the normalized spinor $\frac{1}{\sqrt{65}}\begin{pmatrix}4\\7\end{pmatrix}$. What is the probability that a measurement of S_y yields the value $-\hbar/2$?

Solution:
According to the wordings of the problem, S_y should have one of its eigenvalue equals to $-\hbar/2$. However, the given spinor is in S_z representation. Hence, to solve this problem, we just need to expand the given spinor in terms of the eigenvectors of S_y.

In S_z representation,

$$S_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{\hbar}{2}$$

Let λ be the eigenvalues,

$$\det\begin{pmatrix}-\lambda & -\frac{\hbar}{2}i\\ \frac{\hbar}{2}i & -\lambda\end{pmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$$

$$\Rightarrow \quad \lambda = \pm\frac{\hbar}{2} \qquad \text{as expected.}$$

Now figure out the eigenvectors in S_z representation.

$$\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}u\\ v\end{pmatrix} \quad \Rightarrow \quad \begin{cases}-iv = u\\ iu = v\end{cases}$$

Let $u = 1, v = i$

$\therefore \chi_+ = \begin{pmatrix}1\\ i\end{pmatrix}$ and it has a magnitude of $\sqrt{2}$. Normalizing it, we have

$$\chi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}u\\ v\end{pmatrix} \quad \Rightarrow \quad \begin{cases}-iv = -u\\ iu = -v\end{cases}$$

Let $v = 1, u = i$

$\therefore \chi_- = \begin{pmatrix}i\\ 1\end{pmatrix}$ and it has a magnitude of $\sqrt{2}$. Normalizing it, we have

$$\chi_- = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\ 1\end{pmatrix}$$

Therefore we can construct the transformation matrix (using our "representation tagger") :

$$U = S_z \updownarrow \overset{S_y}{\longleftrightarrow} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

U should be unitary :

$$U^{-1} = U^{+} = S_y \updownarrow \overset{S_z}{\longleftrightarrow} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

We need to transform the given state vector ψ from S_z representation to S_y representation :

$$S_y \ \psi = S_y \updownarrow \overset{S_z}{\longleftrightarrow} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \ S_z \updownarrow \frac{1}{\sqrt{65}}\begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$= S_y \updownarrow \frac{1}{\sqrt{130}}\begin{pmatrix} 4-7i \\ -4i+7 \end{pmatrix}$$

$\therefore$ Probability that a measurement of S_y yields the value $-\frac{\hbar}{2}$

$$= \left| \frac{-4i+7}{\sqrt{130}} \right|^2$$

$$= \frac{16+49}{130}$$

$$= \frac{65}{130}$$

$$= \frac{1}{2}$$