Gasiorowicz 3rd edition Feb 6, 2008

P2-11.

Suppose the V(x) is complex. Obtain an expression for

$$\frac{\partial P(x,t)}{\partial t} \text{ and } \frac{d}{dt}\int_{-\infty}^{\infty} dx\, P(x,t)$$

For absorption of particles the last quantity must be negative (since particle disappear, the probability of their being anywhere decreases). What does this tell us about the imaginary part of V(x)?

Solution:

Let $V(x) = V' + iV''$ where V' is its real part and V'' is its imaginary part..

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial t}(\psi^*\psi) = \left(\frac{\partial}{\partial t}\psi^*\right)\psi + \psi^*\frac{\partial}{\partial t}\psi$$

Schrödinger equation :

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + (V'+iV'')\psi \Rightarrow -i\hbar\frac{\partial}{\partial t}\psi^* = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi^* + (V'-iV'')\psi^*$$

$$\therefore \frac{\partial}{\partial t}P(x,t) = \frac{1}{-i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi^* + (V'-iV'')\psi^*\right)\psi + \psi^*\cdot\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + (V'+iV'')\psi\right)$$

$$= \frac{1}{i\hbar}\left(\frac{\hbar^2}{2m}\right)\left[\left(\frac{\partial^2}{\partial x^2}\psi^*\right)\psi - \psi^*\frac{\partial^2}{\partial x^2}\psi\right] + \frac{2V''}{\hbar}\psi^*\psi$$

$$= -\frac{\partial}{\partial x}\left\{\left(\frac{\hbar}{2im}\right)\left[\psi^*\frac{\partial}{\partial x}\psi - \left(\frac{\partial}{\partial x}\psi^*\right)\psi\right]\right\} + \frac{2V''}{\hbar}|\psi|^2$$

$$= -\frac{\partial}{\partial x}j(x,t) + \frac{2V''}{\hbar}|\psi|^2$$

Integrating this with repect to x,

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\, P(x,t) = \int_{-\infty}^{\infty}\left[-\frac{\partial}{\partial x}j(x,t) + \frac{2V''}{\hbar}|\psi|^2\right]dx$$

$$= \int_{-\infty}^{\infty} -\frac{\partial}{\partial x}j(x,t)dx + \int_{-\infty}^{\infty}\frac{2V''}{\hbar}|\psi|^2\,dx$$

$$= \left[j(-\infty,t) - j(\infty,t)\right] + \int_{-\infty}^{\infty}\frac{2V''}{\hbar}|\psi|^2\,dx$$

$\therefore$ Probability does not conserve if the potential is complex. The imaginary part of the potential acts as a probability source (or sink) in the region underconsideration.