Gasiorowicz 3rd edition Feb 6, 2008

P2-16.

Consider the wave function

$$\psi(x) = (\alpha/\pi)^{1/4} \exp(-\alpha x^2/2)$$

Calculate $<x^n>$ for n=1, 2. Can you quickly write down the result for $<x^{17}>$?

Solution:

$$< x^n > = \int_{-\infty}^{\infty} \psi^*(x) x^n \psi(x)\, dx$$

$$\psi(x) = (\alpha/\pi)^{1/4} \exp(-\alpha x^2/2) = \psi^*(x)$$

$$\therefore < x > = \int_{-\infty}^{\infty} (\alpha/\pi)^{1/4} \exp(-\alpha x^2/2) \cdot x \cdot (\alpha/\pi)^{1/4} \exp(-\alpha x^2/2)\, dx$$

$$= (\alpha/\pi)^{1/2} \int_{-\infty}^{\infty} x \exp(-\alpha x^2)\, dx$$

$$= \underline{\underline{0}} \qquad \text{because } x \exp(-\alpha x^2) \text{ is an odd function.}$$

$$< x^2 > = \int_{-\infty}^{\infty} (\alpha/\pi)^{1/4} \exp(-\alpha x^2/2) \cdot x^2 \cdot (\alpha/\pi)^{1/4} \exp(-\alpha x^2/2)\, dx$$

$$= (\alpha/\pi)^{1/2} \int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\, dx \qquad ---(*)$$

$$\text{Integration by parts: } d\left[x \exp(-\alpha x^2)\right] = \exp(-\alpha x^2)\, dx - 2\alpha x^2 \exp(-\alpha x^2)\, dx$$

$$\therefore \int_{-\infty}^{\infty} d\left[x \exp(-\alpha x^2)\right] = \int_{-\infty}^{\infty} \exp(-\alpha x^2)\, dx - 2\alpha \int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\, dx$$

$$\Rightarrow \left[x \exp(-\alpha x^2)\right]_{-\infty}^{\infty} = \sqrt{\frac{\pi}{\alpha}} - 2\alpha \int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\, dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

Substitute this into (*):

$$< x^2 > = \sqrt{\frac{\alpha}{\pi}} \cdot \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} = \underline{\underline{\frac{1}{2\alpha}}}$$

$$< x^{17} > = \int_{-\infty}^{\infty} (\alpha/\pi)^{1/4} \exp(-\alpha x^2/2) \cdot x^{17} \cdot (\alpha/\pi)^{1/4} \exp(-\alpha x^2/2)\, dx$$

$$= (\alpha/\pi)^{1/2} \int_{-\infty}^{\infty} x^{17} \exp(-\alpha x^2)\, dx$$

$$= \underline{\underline{0}} \qquad \text{because } x^{17} \exp(-\alpha x^2) \text{ is an odd function.}$$