Gasiorowicz 3rd edition Feb 6, 2008

P2-17.

Calculate $\phi(p)$ for the wave function in problem 16. Calculate $<p^n>$ for n=1, 2.

Solution:

Method 1: Do the calculations in p-representation.

$$\psi(x) = (\alpha/\pi)^{1/4} \exp(-\alpha x^2/2) = \psi^*(x)$$

$$\therefore \phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-i\frac{p}{\hbar}x} \psi(x)dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-i\frac{p}{\hbar}x} (\alpha/\pi)^{1/4} \exp(-\alpha x^2/2)dx$$

$$= \frac{(\alpha/\pi)^{1/4}}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp(-\frac{\alpha x^2}{2} - i\frac{p}{\hbar}x)dx$$

$$= \frac{(\alpha/\pi)^{1/4}}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp\left[-\left(\sqrt{\frac{\alpha}{2}}\, x + i\frac{1}{\sqrt{2\alpha}}\frac{p}{\hbar}\right)^2 - \frac{1}{2\alpha}\left(\frac{p}{\hbar}\right)^2 \right]dx$$

$$= \frac{(\alpha/\pi)^{1/4}}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\alpha}\left(\frac{p}{\hbar}\right)^2} \int_{-\infty}^{\infty} \exp\left[-\left(\sqrt{\frac{\alpha}{2}}\, x + i\frac{1}{\sqrt{2\alpha}}\frac{p}{\hbar}\right)^2 \right]dx$$

$$= \frac{(\alpha/\pi)^{1/4}}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\alpha}\left(\frac{p}{\hbar}\right)^2} \cdot \sqrt{\frac{2\pi}{\alpha}}$$

$$= \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha\hbar}} e^{-\frac{1}{2\alpha}\left(\frac{p}{\hbar}\right)^2}$$

In p-representation, operator $\hat{p}$ is just p

$$\therefore <p> = \int_{-\infty}^{\infty} \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha\hbar}} e^{-\frac{1}{2\alpha}\left(\frac{p}{\hbar}\right)^2} \cdot p \cdot \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha\hbar}} e^{-\frac{1}{2\alpha}\left(\frac{p}{\hbar}\right)^2} dx$$

$$= \frac{(\alpha/\pi)^{1/2}}{\alpha\hbar} \int_{-\infty}^{\infty} p\, e^{-\frac{1}{\alpha}\left(\frac{p}{\hbar}\right)^2} dp$$

$$= \underline{\underline{0}} \qquad \text{because } p\, e^{-\frac{1}{\alpha}\left(\frac{p}{\hbar}\right)^2} \text{ is an odd function.}$$

$$<p^2> = \int_{-\infty}^{\infty} \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha\hbar}} e^{-\frac{1}{2\alpha}\left(\frac{p}{\hbar}\right)^2} \cdot p^2 \cdot \frac{(\alpha/\pi)^{1/4}}{\sqrt{\alpha\hbar}} e^{-\frac{1}{2\alpha}\left(\frac{p}{\hbar}\right)^2} dx$$

$$= \frac{(\alpha/\pi)^{1/2}}{\alpha\hbar} \int_{-\infty}^{\infty} p^2\, e^{-\frac{1}{\alpha}\left(\frac{p}{\hbar}\right)^2} dp \qquad ---(*)$$

Integration by parts: $d\left(p\ e^{-\frac{1}{\alpha}\left(\frac{p}{\hbar}\right)^2}\right) = e^{-\frac{1}{\alpha}\left(\frac{p}{\hbar}\right)^2}\ dp - 2\frac{p^2}{\alpha\hbar^2}\ e^{-\frac{1}{\alpha}\left(\frac{p}{\hbar}\right)^2}\ dx$

$$\int_{-\infty}^{\infty} d\left(p\ e^{-\frac{1}{\alpha}\left(\frac{p}{\hbar}\right)^2}\right) = \int_{-\infty}^{\infty} e^{-\frac{1}{\alpha}\left(\frac{p}{\hbar}\right)^2}\ dp - \int_{-\infty}^{\infty} 2\frac{p^2}{\alpha\hbar^2}\ e^{-\frac{1}{\alpha}\left(\frac{p}{\hbar}\right)^2}\ dx$$

$$\Rightarrow \left[p\ e^{-\frac{1}{\alpha}\left(\frac{p}{\hbar}\right)^2}\right]_{-\infty}^{\infty} = \sqrt{\alpha\hbar^2\pi} - \frac{2}{\alpha\hbar^2}\int_{-\infty}^{\infty} p^2\ e^{-\frac{1}{\alpha}\left(\frac{p}{\hbar}\right)^2}\ dp$$

$$\Rightarrow \int_{-\infty}^{\infty} p^2\ e^{-\frac{1}{\alpha}\left(\frac{p}{\hbar}\right)^2}\ dp = \frac{\alpha\hbar^2}{2}\sqrt{\alpha\hbar^2\pi}$$

Substitute this into (*):

$$< p^2 > = \frac{(\alpha/\pi)^{1/2}}{\alpha\hbar}\cdot\frac{\alpha\hbar^2}{2}\sqrt{\alpha\hbar^2\pi} = \underline{\underline{\frac{\alpha\hbar^2}{2}}}$$

Method 2: Do the calculations in x-representation.

$$\psi(x) = (\alpha/\pi)^{1/4}\ \exp(-\alpha\ x^2/2) = \psi^*(x)$$

In x - represention, momentum operator is $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

$$\therefore < p > = \int_{-\infty}^{\infty} (\alpha/\pi)^{1/4}\ \exp(-\alpha\ x^2/2)\cdot\left(-i\hbar\frac{\partial}{\partial x}\right)\cdot(\alpha/\pi)^{1/4}\ \exp(-\alpha\ x^2/2)\ dx$$

$$= -i\hbar(\alpha/\pi)^{1/2}\int_{-\infty}^{\infty} \exp(-\alpha\ x^2/2)\frac{\partial}{\partial x}\exp(-\alpha\ x^2/2)dx$$

$$= -i\hbar(\alpha/\pi)^{1/2}\int_{-\infty}^{\infty} \exp(-\alpha\ x^2/2)\cdot -\alpha x\ \exp(-\alpha\ x^2/2)\ dx$$

$$= i\hbar\alpha(\alpha/\pi)^{1/2}\int_{-\infty}^{\infty} x\ \exp(-\alpha\ x^2)\ dx$$

$= \underline{\underline{0}}$ because $x\ \exp(-\alpha\ x^2)$ is an odd function.

$$< p^2 > = \int_{-\infty}^{\infty} (\alpha/\pi)^{1/4}\ \exp(-\alpha\ x^2/2)\cdot\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\cdot(\alpha/\pi)^{1/4}\ \exp(-\alpha\ x^2/2)\ dx$$

$$= -\hbar^2(\alpha/\pi)^{1/2}\int_{-\infty}^{\infty} \exp(-\alpha\ x^2/2)\cdot\frac{\partial}{\partial x}[-\alpha x\ \exp(-\alpha\ x^2/2)]\ dx$$

$$= -\hbar^2\alpha(\alpha/\pi)^{1/2}\int_{-\infty}^{\infty} \exp(-\alpha\ x^2/2)[-\exp(-\alpha\ x^2/2) + \alpha\ x^2\ \exp(-\alpha\ x^2/2)]dx$$

$$< p^2 > = -\hbar^2 \alpha (\alpha/\pi)^{1/2} \int_{-\infty}^{\infty} \exp(-\alpha x^2)[-1 + \alpha x^2] dx$$

$$= -\hbar^2 \alpha (\alpha/\pi)^{1/2} \left[-\int_{-\infty}^{\infty} \exp(-\alpha x^2) dx + \alpha \int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2) dx \right] \qquad ---(**)$$

Integration by parts,

$$d\left[x \exp(-\alpha x^2)\right] = \exp(-\alpha x^2)\, dx - 2\alpha x^2 \exp(-\alpha x^2)\, dx$$

$$\therefore \int_{-\infty}^{\infty} d\left[x \exp(-\alpha x^2)\right] = \int_{-\infty}^{\infty} \exp(-\alpha x^2)\, dx - 2\alpha \int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\, dx$$

$$\Rightarrow \left[x \exp(-\alpha x^2)\right]_{-\infty}^{\infty} = \sqrt{\frac{\pi}{\alpha}} - 2\alpha \int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\, dx = \frac{1}{2\alpha} \sqrt{\frac{\pi}{\alpha}}$$

Substitute into (**):

$$< p^2 > = -\hbar^2 \alpha (\alpha/\pi)^{1/2} \left[-\sqrt{\frac{\pi}{\alpha}} + \alpha \cdot \frac{1}{2\alpha} \sqrt{\frac{\pi}{\alpha}} \right]$$

$$= \hbar^2 \alpha (\alpha/\pi)^{1/2} \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}$$

$$= \underline{\underline{\frac{\hbar^2 \alpha}{2}}}$$

Both methods give the same result.