Gasiorowicz 3rd edition Jan 25, 2008

P2-2.

The relation between the wavelength λ and the frequency ν in a wave guide is given by

$$\lambda = \frac{c}{\sqrt{\nu^2 - \nu_0^2}}$$

What is the group velocity of such wave?

Solution:

$$\lambda = \frac{c}{\sqrt{\nu^2 - \nu_0^2}}$$

$$k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k}$$

$$\omega = 2\pi\nu \Rightarrow \nu = \frac{\omega}{2\pi}$$

$$\therefore \frac{2\pi}{k} = \frac{c}{\sqrt{\left(\frac{\omega}{2\pi}\right)^2 - \left(\frac{\omega_0}{2\pi}\right)^2}} \Rightarrow \frac{1}{k} = \frac{c}{\sqrt{\omega^2 - {\omega_0}^2}}$$

$$\Rightarrow \frac{1}{k} = c\left(\omega^2 - {\omega_0}^2\right)^{-\frac{1}{2}}$$

$$\Rightarrow -\frac{1}{k^2}\,dk = -\frac{1}{2}c\left(\omega^2 - {\omega_0}^2\right)^{-\frac{3}{2}} \cdot 2\omega\,d\omega$$

$$\Rightarrow \frac{1}{k^2}\,dk = \frac{c\omega}{\left(\omega^2 - {\omega_0}^2\right)^{\frac{3}{2}}}\,d\omega$$

$$\Rightarrow \frac{d\omega}{dk} = \frac{\left(\omega^2 - {\omega_0}^2\right)^{\frac{3}{2}}}{c\omega\,k^2}$$

$$\text{But } \frac{1}{k} = c\left(\omega^2 - {\omega_0}^2\right)^{-\frac{1}{2}} \Rightarrow \left(\omega^2 - {\omega_0}^2\right)^{\frac{1}{2}} = ck \Rightarrow \left(\omega^2 - {\omega_0}^2\right)^{\frac{3}{2}} = c^3k^3$$

$$\therefore \quad v_g = \frac{d\omega}{dk} = \frac{c^3k^3}{c\omega\,k^2}$$

$$\Rightarrow v_g = \frac{c^2k}{\omega}$$

$$\Rightarrow v_g = c^2\frac{2\pi}{\lambda}\frac{1}{2\pi\nu}$$

$$\Rightarrow v_g = \frac{c^2}{\underline{\underline{\lambda\nu}}}$$