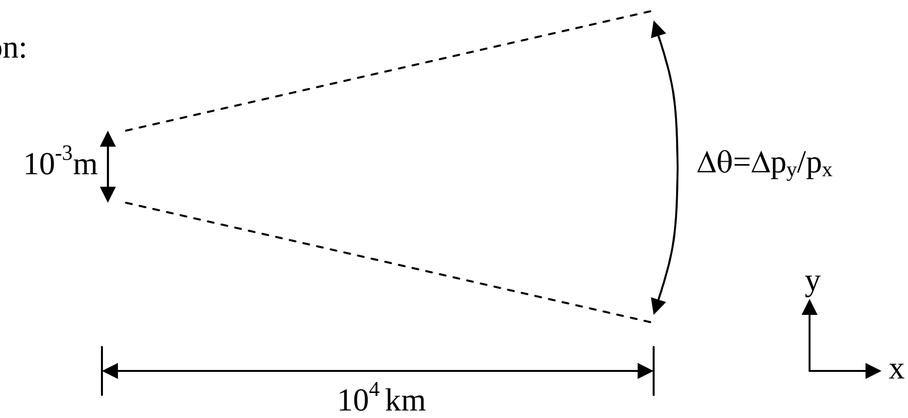
Gasiorowicz 3rd edition
P2-6.
A beam of electrons is to be fired over a distance of 10^4 km. If the size of the initial particle is 10^{-3}m, what will be its size upon arrival, if its kinetic energy is (a) 13.6 eV; (b) 100MeV? [Caution: The relation between kinetic energy (K.E.) and momentum is not always K.E. = $p^2/2m$!]

Solution:



For both cases,

$$\Delta y = 10^{-3}\ \text{m}$$

$$\Delta p_y \Delta y \approx \frac{\hbar}{2} \Rightarrow 10^{-3}\Delta p_y = \frac{1.055\times10^{-34}}{2}$$

$$\Rightarrow \Delta p_y = 5.275\times10^{-32}\ \text{Ns}$$

$$m_0 = 9.1095\times10^{-31}\ \text{kg}$$

$$m_0c^2 = 9.1095\times10^{-31}\times\left(3\times10^8\right)^2 = 8.199\times10^{-14}\ \text{J}$$

(a) $\text{K.E.} = 13.6\ \text{eV} = 13.6\times1.602\times10^{-19} = 2.179\times10^{-18}\ \text{J}$

$\therefore$ K.E. $<< m_0c^2$ and this is a classical case.

$$\text{K.E.} = \frac{p_x^2}{2m} \Rightarrow p_x^2 = 2.179\times10^{-18}\times2\times9.1095\times10^{-31} = 3.970\times10^{-48}$$

$$\Rightarrow p_x = 1.992\times10^{-24}\ \text{Ns}$$

$$\therefore \Delta\theta = \frac{\Delta p_y}{p_x} = \frac{5.275\times10^{-32}}{1.992\times10^{-24}} = 2.648\times10^{-8}\ \text{rad}$$

$\therefore$ Spread of the beam upon arrival $= 2.648\times10^{-8}\times10^7\ \text{m} = \underline{\underline{0.26\text{m}}}$

(b) $\text{K.E.} = 100\text{ MeV} = 100\times10^{6}\times1.602\times10^{-19} = 1.602\times10^{-11}\text{ J}$

$\therefore\ \text{K.E.} > m_0c^2$ and this is a relativistic case.

$$\text{K.E.} = E - m_0c^2 = \sqrt{p^2c^2 + m_0{}^2c^4} - m_0c^2$$

$$\Rightarrow \sqrt{p^2c^2 + m_0{}^2c^4} = \text{K.E.} + m_0c^2 = 1.602\times10^{-11} + 8.199\times10^{-14}$$

$$= 1.610\times10^{-11}$$

$$\Rightarrow p^2c^2 + m_0{}^2c^4 = (1.610\times10^{-11})^2 = 2.593\times10^{-22}$$

$$\Rightarrow p^2c^2 = 2.593\times10^{-22} - m_0{}^2c^4 = 2.593\times10^{-22} - (8.199\times10^{-14})^2$$

$$= 2.593\times10^{-22}$$

$$\Rightarrow pc = \sqrt{2.593\times10^{-22}} = 1.602\times10^{-11} \qquad (\sim \text{K.E. because K.E.} >> m_0c^2)$$

$$\Rightarrow p = \frac{1.602\times10^{-11}}{3\times10^{8}} = 5.367\times10^{-20}\ \text{Ns}$$

$$\therefore p_x = 5.367\times10^{-20}\ \text{Ns}$$

$$\therefore\ \Delta\theta = \frac{\Delta p_y}{p_x} = \frac{5.275\times10^{-32}}{5.367\times10^{-20}} = 9.828\times10^{-13}\text{ rad}$$

$\therefore$ Spread of the beam upon arrival $= 9.828\times10^{-13}\times10^{7}$ m

$$= 9.828\times10^{-6}\text{ m} << \text{initial size}$$

$\therefore$ Size of the beam upon arrival will be the same as the initial size, 10^{-3} m