Gasiorowicz 3rd edition   Feb 17, 2008

P3-1.

1. You are given the following operators:

(a) $O_1\psi(x) = x^3\psi(x)$; (b) $O_2\psi(x) = x(d/dx)\ \psi(x)$ (c) $O_3\psi(x) = \lambda\ \psi^*(x)$;

(d) $O_4\psi(x) = e^{\psi(x)}$; (e) $O_5\psi(x) = [d\psi(x)/dx]+a$ (f) $O_6\psi(x) = \int_{-\infty}^{x} dx'(\psi(x')x')$;

Which of these are linear operators?

Solution:

For a linear operator L, we require $L[a\psi_1(x)+b\psi_2(x)]= aL[\psi_1(x)]+bL[\psi_2(x)]$, so let us test the given operators one by one:

$$
\begin{aligned}
\text{(a) } O_1[a\psi_1(x)+b\psi_2(x)] &= x^3\,[a\psi_1(x)+b\psi_2(x)] \\
&= a[x^3\psi_1(x)]+b[x^3\psi_2(x)] \\
&= a\,O_1\psi_1(x)+bO_1\psi_2(x)
\end{aligned}
$$

So O_1 is a linear operator.

$$
\begin{aligned}
\text{(b) } O_2[a\psi_1(x)+b\psi_2(x)] &= x\frac{d}{dx}[a\psi_1(x)+b\psi_2(x)] \\
&= a[x\frac{d}{dx}\psi_1(x)]+b[x\frac{d}{dx}\psi_2(x)] \\
&= a\,O_2\psi_1(x)+bO_2\psi_2(x)
\end{aligned}
$$

So O_2 is a linear operator.

$$
\begin{aligned}
\text{(c) } O_3[a\psi_1(x)+b\psi_2(x)] &= \lambda\,[a\psi_1(x)+b\psi_2(x)]^* \\
&= a^*[\lambda\psi_1{}^*(x)]+b^*[\lambda\psi_2{}^*(x)] \\
&= a^*O_3\psi_1(x)+b^*O_3\psi_2(x) \\
&\neq a\,O_3\psi_1(x)+bO_3\psi_2(x) \quad \text{if a or b are not real}
\end{aligned}
$$

So O_3 is NOT a linear operator.

$$
\begin{aligned}
\text{(d) } O_2[a\psi_1(x)+b\psi_2(x)] &= e^{\,[a\psi_1(x)+b\psi_2(x)]} \\
&= [e^{\psi_1(x)}]^a \times [e^{\psi_2(x)}]^b \\
&\neq ae^{\psi_1(x)}+be^{\psi_2(x)} \\
&\neq a\,O_4\psi_1(x)+bO_4\psi_2(x)
\end{aligned}
$$

So O_4 is NOT a linear operator.

$$(e)\ O_5[b\psi_1(x)+c\psi_2(x)]=\frac{d}{dx}[b\psi_1(x)+c\psi_2(x)]+a$$

$$=b[\frac{d}{dx}\psi_1(x)]+c[\frac{d}{dx}\psi_2(x)]+a$$

$$\neq b\left([\frac{d}{dx}\psi_1(x)]+a\right)+c\left([\frac{d}{dx}\psi_2(x)]+a\right)$$

$$\neq b\,O_5\psi_1(x)+cO_5\psi_2(x)$$

So O_5 is NOT a linear operator.

$$(f)\ O_6[a\psi_1(x)+b\psi_2(x)]=\int_{-\infty}^{x} x'[a\psi_1(x')+b\psi_2(x')]dx'$$

$$=a\int_{-\infty}^{x} x'\psi_1(x')dx'+b\int_{-\infty}^{x} x'\psi_2(x')dx'$$

$$=a\,O_6\psi_1(x)+bO_6\psi_2(x)$$

So O_6 is a linear operator.