Gasiorowicz 3rd edition Feb 17, 2008

P3-10.

Consider the wave function of Problem 9.

(a) Calculate the probability that an energy measurement yields the energy eigenvalue associated with the particular value of n.

(b) Use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

to show that the probability add up to one.

[P3-9:

A particle is known to be localized in the left half of a box with sides at x=±a/2. woth wave function

$$\psi(x) = \begin{cases} = \sqrt{\frac{2}{a}} & \frac{-a}{2} < x < 0 \\ = 0 & 0 < x < \frac{a}{2} \end{cases}$$

(a) Will the particle remain localized at later times?

(b) Calculate the probability that an energy measurement yields the ground state energy; the energy of the first excited state.]

Solution:

$$\text{(a)}\quad |\psi\rangle = \begin{cases} = \sqrt{\dfrac{2}{a}} & \dfrac{-a}{2} < x < 0 \\ = 0 & 0 < x < \dfrac{a}{2} \end{cases}$$

$$\text{If } |\psi\rangle = \sum_{i=1}^{\infty} a_n\, |u_n(x)\rangle$$

$$a_n = \langle u_n(x)|\psi\rangle = \int_{-a/2}^{a/2} u_n^*(x)\,\psi(x)dx$$

$$= \sqrt{\frac{2}{a}} \int_{-a/2}^{0} u_n^*(x)\,dx$$

$$= \begin{cases} \sqrt{\dfrac{2}{a}} \displaystyle\int_{-a/2}^{0} \sqrt{\dfrac{2}{a}} \cos\dfrac{n\pi x}{a}\,dx & \text{odd n} \\ \sqrt{\dfrac{2}{a}} \displaystyle\int_{-a/2}^{0} \sqrt{\dfrac{2}{a}} \sin\dfrac{n\pi x}{a}\,dx & \text{even n} \end{cases}$$

$$= \begin{cases} \left[\dfrac{2}{a}\dfrac{a}{n\pi}\sin\dfrac{n\pi x}{a}\right]_{-a/2}^{0} & \text{odd n} \\ \left[-\dfrac{2}{a}\dfrac{a}{n\pi}\cos\dfrac{n\pi x}{a}\right]_{-a/2}^{0} & \text{even n} \end{cases}$$

$$= \begin{cases} (-1)^{\frac{n-1}{2}}\,\dfrac{2}{n\pi} & \text{odd n} \\ \left[-1+(-1)^{\frac{n}{2}}\right]\dfrac{2}{n\pi} & \text{even n} \end{cases}$$

$$\therefore \quad a_n^{\,2} = \begin{cases} \dfrac{4}{n^2\pi^2} & \text{odd n} \\ \dfrac{16}{n^2\pi^2} & \text{even n but not mutiple of } 4 \\ 0 & \text{multiple of } 4 \end{cases}$$

(b) Note that there is a misprint in the text book.

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$ is not correct. Instead, only the odd terms add up to $\pi^2/8$. i.e.,

$$\sum_{n=1}^{odd} \frac{1}{n^2} = \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n^{\,2} &= \sum_{n\ \text{odd}} \frac{4}{n^2\pi^2} + \sum_{\substack{\text{n even but not}\\ \text{multiple of 4}}} \frac{16}{n^2\pi^2} \\
&= \sum_{m=1}^{\infty} \frac{4}{(2m+1)^2\pi^2} + \sum_{k=1}^{\infty} \frac{16}{(4k+2)^2\pi^2} \\
&= \sum_{m=1}^{\infty} \frac{4}{(2m+1)^2\pi^2} + \sum_{k=1}^{\infty} \frac{16}{4(2k+1)^2\pi^2} \\
&= \sum_{m=1}^{\infty} \frac{4}{(2m+1)^2\pi^2} + \sum_{k=1}^{\infty} \frac{4}{(2k+1)^2\pi^2} \\
&= \sum_{m=1}^{\infty} \frac{8}{(2m+1)^2\pi^2} \\
&= \frac{8}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m+1)^2} \\
&= \frac{8}{\pi^2}\frac{\pi^2}{8} \\
&= 1
\end{aligned}$$