Gasiorowicz 3rd edition Feb 6, 2008

P3-11.

The eigenfunctions for a potential of the form

$$V(x) = \infty \qquad 0<a;\; x>a$$
$$= 0 \qquad 0 < x < a$$

are of the form

$$u_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} x$$

Suppose a particle in the preceding potential has an initial normalized wave function of the form

$$\psi(x,0) = A\left(\sin \frac{\pi x}{a}\right)^5 \qquad \leftarrow \text{ Note misprint in text book}$$

(a) What is the form of $\psi(x,t)$?

(b) Calculate A without doing the integral $\int d\theta \sin^{10}\theta$.

(c) What is the probability that an energy measurement yields E_3, where $E_n = n^2\pi^2\hbar^2/2ma^2$?

{*Hint*: Expand $([e^{i\theta} - e^{-i\theta}]/2i)^5$ in a power series $e^{5i\theta} + \ldots - e^{-5i\theta}$ and recombine into a series of terms involving sin 5θ and so on}.

Solution:

(a)

$$\psi(x,0) = A\left(\sin \frac{\pi x}{a}\right)^5 = A\left(\frac{e^{i\frac{\pi x}{a}} - e^{-i\frac{\pi x}{a}}}{2i}\right)^5 = \frac{A}{32i}\left(e^{i\frac{5\pi x}{a}} - 5e^{\frac{3\pi x}{a}} + 10e^{\frac{\pi x}{a}} - 10e^{-\frac{\pi x}{a}} + 5e^{-\frac{3\pi x}{a}} - e^{-\frac{5\pi x}{a}}\right)$$

$$= \frac{A}{16}\left(\frac{e^{i\frac{5\pi x}{a}} - e^{-i\frac{5\pi x}{a}}}{2i} + 5\frac{e^{\frac{3\pi x}{a}} - e^{-\frac{3\pi x}{a}}}{2i} + 10\frac{e^{\frac{\pi x}{a}} - e^{-\frac{\pi x}{a}}}{2i}\right)$$

$$= \frac{A}{16}\left(\sin\frac{5\pi x}{a} + 5\sin\frac{3\pi x}{a} + 10\sin\frac{\pi x}{a}\right)$$

$$= \frac{A}{16}\sqrt{\frac{a}{2}}\left[u_5(x) + 5u_3(x) + 10u_1(x)\right]$$

$$\therefore \psi(x,t) = \frac{A}{16}\sqrt{\frac{a}{2}}\left[u_5(x)e^{-i\frac{E_5}{\hbar}t} + 5u_3(x)e^{-i\frac{E_3}{\hbar}t} + 10u_1(x)e^{-i\frac{E_1}{\hbar}t}\right] \qquad \text{with } E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$= \frac{A}{16}\left[\sin\frac{5\pi x}{a}e^{-i\frac{25\pi^2\hbar}{2ma^2}t} + 5\sin\frac{3\pi x}{a}e^{-i\frac{9\pi^2\hbar}{2ma^2}t} + 10\sin\frac{\pi x}{a}e^{-i\frac{\pi^2\hbar}{2ma^2}t}\right]$$

With $A = \frac{16}{3}\sqrt{\frac{1}{7a}}$ from part (b),

$$\underline{\underline{\psi(x,t) = \frac{1}{3\sqrt{7a}}\left[\sin\frac{5\pi x}{a}e^{-i\frac{25\pi^2\hbar}{2ma^2}t} + 5\sin\frac{3\pi x}{a}e^{-i\frac{9\pi^2\hbar}{2ma^2}t} + 10\sin\frac{\pi x}{a}e^{-i\frac{\pi^2\hbar}{2ma^2}t}\right]}}$$

(b)

From part (a) :

$$|\psi(x,0)> = \frac{A}{16}\sqrt{\frac{a}{2}}\left[|u_5(x)> + 5|u_3(x)> + 10|u_1(x)>\right]$$

$$\Rightarrow <\psi(x,0)|\psi(x,0)> = \frac{A^2}{256}\frac{a}{2}\left[<u_5(x)|u_5(x)> + 25<u_3(x)|u_3(x)> + 100<u_1(x)|u_1(x)>\right]$$

$$\Rightarrow 1 = \frac{A^2}{256}\frac{a}{2}\times 126$$

$$\Rightarrow 1 = \frac{A^2 a}{256}\times 63$$

$$\Rightarrow A^2 = \frac{256}{a}\times\frac{1}{63}$$

$$\Rightarrow A = \underline{\underline{\frac{16}{3}\sqrt{\frac{1}{7a}}}}$$

(c)

From part (a) abd (b) :

$$|\psi(x,0)> = \frac{A}{16}\sqrt{\frac{a}{2}}\left[|u_5(x)> + 5|u_3(x)> + 10|u_1(x)>\right]$$

$$= \frac{1}{3\sqrt{14}}\left[|u_5(x)> + 5|u_3(x)> + 10|u_1(x)>\right] \qquad (\because A = \frac{16}{3}\sqrt{\frac{1}{7a}})$$

$\therefore$ Probability for the system to be at $n = 3$ state is $\left(\frac{5}{3\sqrt{14}}\right)^2 = \underline{\underline{\frac{25}{136} \sim 19.8\%}}$