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P3-12.

A particle is in the ground state of a box with sides at x=±a. Very suddeny the sides of the box are moved to x=±b (b>a). What is the probability that the particle will be found in the ground state for the new potential? What is the probability that it will be found in the first excited state? In the latter case, the simple answer has a simple explanation. What is it?

Solution:

(a)

For a box potential symmetric about x = 0,

$$u_n(x) = \begin{cases} \sqrt{\frac{1}{a}}\cos\left(\frac{n\pi x}{2a}\right) & n = 1,3,5,\ldots.. \\ \sqrt{\frac{1}{a}}\sin\left(\frac{n\pi x}{2a}\right) & n = 2,4,6,\ldots. \end{cases}$$

Since the particle is initially in the ground state,

$$\therefore \psi(x) = u_1(x) = \sqrt{\frac{1}{a}}\cos\left(\frac{\pi x}{2a}\right)$$

If the walls are moved from ± a to ± b, the eigienfunctions will become

$$w_n(x) = \begin{cases} \sqrt{\frac{1}{b}}\cos\left(\frac{n\pi x}{2b}\right) & n = 1,3,5,\ldots.. \\ \sqrt{\frac{1}{b}}\sin\left(\frac{n\pi x}{2b}\right) & n = 2,4,6,\ldots. \end{cases}$$

∴ Probability that the particle will be fiound in the ground state for the new potential

$$= |\langle w_1 | \psi \rangle|^2$$

$$= \left| \int_{-a}^{a} \sqrt{\frac{1}{b}}\cos\left(\frac{\pi x}{2b}\right)\sqrt{\frac{1}{a}}\cos\left(\frac{\pi x}{2a}\right)dx \right|^2$$

$$= \frac{1}{ab}\left| \int_{-a}^{a} \frac{1}{2}\left\{\cos\left[\left(\frac{\pi x}{2b}\right)+\left(\frac{\pi x}{2a}\right)\right]+\cos\left[\left(\frac{\pi x}{2b}\right)-\left(\frac{\pi x}{2a}\right)\right]\right\}dx \right|^2$$

$$= \frac{1}{4ab}\left| \int_{-a}^{a} \cos\frac{\pi}{2}\left(\frac{1}{a}+\frac{1}{b}\right)x\,dx + \int_{-a}^{a} \cos\frac{\pi}{2}\left(\frac{1}{a}-\frac{1}{b}\right)x\,dx \right|^2$$

$$= \frac{1}{4ab}\left| \left[\frac{1}{\frac{\pi}{2}\left(\frac{1}{a}+\frac{1}{b}\right)} \sin\frac{\pi}{2}\left(\frac{1}{a}+\frac{1}{b}\right)x \right]_{-a}^{a} + \left[\frac{1}{\frac{\pi}{2}\left(\frac{1}{a}-\frac{1}{b}\right)} \sin\frac{\pi}{2}\left(\frac{1}{a}-\frac{1}{b}\right)x \right]_{-a}^{a} \right|^2$$

$$= \frac{1}{4ab}\left| \frac{2}{\frac{\pi}{2}\left(\frac{1}{a}+\frac{1}{b}\right)} \sin\frac{\pi a}{2}\left(\frac{1}{a}+\frac{1}{b}\right) + \frac{2}{\frac{\pi}{2}\left(\frac{1}{a}-\frac{1}{b}\right)} \sin\frac{\pi a}{2}\left(\frac{1}{a}-\frac{1}{b}\right) \right|^2$$

$$= \frac{4}{ab\pi^2}\left| \frac{ab}{a+b}\sin\left(\frac{\pi a}{2b}+\frac{\pi}{2}\right) + \frac{ab}{b-a}\sin\left(\frac{\pi}{2}-\frac{\pi a}{2b}\right) \right|^2$$

$$= \frac{4}{ab\pi^2}\left(\frac{ab}{a+b} + \frac{ab}{b-a} \right)^2 \cos^2\left(\frac{\pi a}{2b}\right)$$

$$= \frac{4}{ab\pi^2}\frac{a^2b^2 \cdot 4b^2}{(b^2-a^2)^2}\cos^2\left(\frac{\pi a}{2b}\right)$$

$$= \underline{\underline{\frac{16ab^3}{(b^2-a^2)^2\pi^2}\cos^2\left(\frac{\pi a}{2b}\right)}}$$

Note:

When the move is small, $b = a + \varepsilon \quad (\varepsilon \to 0) \Rightarrow a = b - \varepsilon$

$$\text{Probability} = \frac{16ab^3}{(b^2-a^2)^2\pi^2}\cos^2\left(\frac{\pi a}{2b}\right) = \frac{16(b-\varepsilon)b^3}{[b^2-(b-\varepsilon)^2]^2\pi^2}\cos^2\left(\frac{\pi}{2}(1-\frac{\varepsilon}{b})\right)$$

$$\sim \frac{16(b-\varepsilon)b^3}{[2b\varepsilon]^2\pi^2}\sin^2\left(\frac{\pi\varepsilon}{2b}\right)$$

$$\sim \frac{16(b-\varepsilon)b^3}{[2b\varepsilon]^2\pi^2}\cdot\frac{\pi^2\varepsilon^2}{4b^2}$$

$$\sim 1 - \frac{\varepsilon}{b}$$

Probability that the particle will be fiound in the first excited state for the new potential

$$=|< w_2 | \psi >|^2$$

$$=\left|\int_{-a}^{a} \sqrt{\frac{1}{b}}\sin\left(\frac{2\pi x}{2b}\right)\sqrt{\frac{1}{a}}\cos\left(\frac{\pi x}{2a}\right)dx\right|^2$$

$$=\frac{1}{ab}\left|\int_{-a}^{a} \frac{1}{2}\left\{\sin\left[\left(\frac{\pi x}{2b}\right)+\left(\frac{\pi x}{2a}\right)\right]+\sin\left[\left(\frac{\pi x}{2b}\right)-\left(\frac{\pi x}{2a}\right)\right]\right\}dx\right|^2$$

$$=\frac{1}{4ab}\left|\int_{-a}^{a} \sin\frac{\pi}{2}\left(\frac{1}{a}+\frac{1}{b}\right)x\,dx+\int_{-a}^{a} \sin\frac{\pi}{2}\left(\frac{1}{a}-\frac{1}{b}\right)x\,dx\right|^2$$

$$=\frac{1}{4ab}\left|\left[-\frac{1}{\frac{\pi}{2}\left(\frac{1}{a}+\frac{1}{b}\right)}\cos\frac{\pi}{2}\left(\frac{1}{a}+\frac{1}{b}\right)x\right]_{-a}^{a}+\left[-\frac{1}{\frac{\pi}{2}\left(\frac{1}{a}-\frac{1}{b}\right)}\cos\frac{\pi}{2}\left(\frac{1}{a}-\frac{1}{b}\right)x\right]_{-a}^{a}\right|^2$$

$$=\frac{1}{4ab}|0+0|^2 \qquad (\because \cos\alpha a=\cos\alpha(-a))$$

$$=\underline{\underline{0}}$$

It is zero because the wave function is even while the first excited state is odd.