Gasiorowicz 3rd edition Feb 17, 2008

P3-4.

Calculate

$$\Delta x = \sqrt{< x^2 >}$$

For the $u_n(x)$ given in (3-21). Using $<p^2>$ given by (3-26), calculate

$$\Delta p\ \Delta x$$

It is characteristic that for the higher states the uncertainty increases with n.

[Equation (3-21):

$$u_n(x) = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}$$

Equation (3-26):

$$< p^2 >= 2mE_n = \frac{\hbar^2\pi^2 n^2}{a^2} \quad]$$

Solution:

$$< x^2 >= \int_0^a u_n^*(x)x^2u^*(x)dx = \int_0^a \left(\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\right)x^2\left(\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\right)dx$$

$$= \frac{2}{a}\int_0^a x^2 \sin^2\frac{n\pi x}{a}dx$$

$$= \frac{2}{a}\int_0^a x^2\left(\frac{1}{2} - \frac{1}{2}\cos\frac{2n\pi x}{a}\right)dx$$

$$= \frac{a^2}{3} - \frac{1}{a}\int_0^a x^2\cos\alpha x\, dx \qquad \text{where } \alpha = \frac{2n\pi}{a}$$

Note that $\frac{\partial}{\partial\alpha}\cos\alpha x = -x\sin\alpha x$, $\frac{\partial^2}{\partial\alpha^2}\cos\alpha x = -x^2\cos\alpha x$

$$\int x^2\cos\alpha x\, dx = \int -\frac{\partial^2}{\partial\alpha^2}\cos\alpha x\, dx = -\frac{\partial^2}{\partial\alpha^2}\int \cos\alpha x\, dx = -\frac{\partial^2}{\partial\alpha^2}\left(\frac{1}{\alpha}\sin\alpha x\right)$$

$$= \frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha^2}\sin\alpha x - \frac{x}{\alpha}\cos\alpha x\right) = -\frac{2}{\alpha^3}\sin\alpha x + \frac{x}{\alpha^2}\cos\alpha x + \frac{x}{\alpha^2}\cos\alpha x + \frac{x^2}{\alpha}\sin\alpha x$$

$$= -\frac{2}{\alpha^3}\sin\alpha x + \frac{2x}{\alpha^2}\cos\alpha x + \frac{x^2}{\alpha}\sin\alpha x$$

$$\therefore < x^2 >= \frac{a^2}{3} + \frac{1}{a}\left[\frac{2}{\alpha^3}\sin\alpha x - \frac{2x}{\alpha^2}\cos\alpha x - \frac{x^2}{\alpha}\sin\alpha x\right]_0^a$$

$$\sin\alpha a = \sin\frac{2n\pi}{a}\cdot a = 0 \text{ and } \cos\alpha a = \cos\frac{2n\pi}{a}\cdot a = 1$$

$$\therefore \frac{1}{a}\left[\frac{2}{\alpha^3}\sin\alpha x - \frac{2x}{\alpha^2}\cos\alpha x - \frac{x^2}{\alpha}\sin\alpha x\right]_0^a = \frac{1}{a}\left(-\frac{2a}{\alpha^2}\cos\alpha a\right) = -\frac{1}{a}\frac{2a}{\left(\frac{2n\pi}{a}\right)^2} = -\frac{a^2}{2n^2\pi^2}$$

$$\therefore \langle x^2 \rangle = \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}$$

There is a misprint in this problem because u_n given by equation (3-21) is for potential well $0<x<a$ and hence $\langle x\rangle \neq 0$.

If $\langle x \rangle \neq 0$ then $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ but not $\sqrt{\langle x^2 \rangle}$ as indicated by this problem.

For this potential well, $\langle x \rangle = \frac{a}{2}$

$$\therefore \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\left(\frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}\right) - \left(\frac{a}{2}\right)^2} = a\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

Now $\langle p^2 \rangle = \frac{\hbar^2\pi^2 n^2}{a^2}$, with $\langle p \rangle = 0$, $\therefore \Delta p = \sqrt{\langle p^2 \rangle} = \frac{\hbar\pi n}{a}$

$$\therefore \Delta x \Delta p = a\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}\cdot\frac{\hbar\pi n}{a} = \hbar\pi n\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

For $n = 1$, $\Delta x\Delta p = 0.568\hbar$

$n = 2$, $\Delta x\Delta p = 1.670\hbar$

$n = 3$, $\Delta x\Delta p = 2.627\hbar$

For large n, $\Delta x\Delta p \sim \hbar\pi n\sqrt{\frac{1}{12}}$ or $0.907n\hbar$