Gasiorowicz 3rd edition Feb 17, 2008

P3-5.

Calculate an electron of mass m=0.9×10^{-30} kg in an infinite box of dimension a=10^{-9} m.

(a) What is the energy difference between the ground state and the first excited state.? Express your answer in eV.

(b) Suppose the transition from the state n=2 to the state n=1 is accompanied by the emission of a photon, as given by the Bohr rule. What is the wavelength of the emitted photon?

Solution:

(a) $E_n = \dfrac{\hbar^2 k^2}{2m}$

$$\text{For } n = 1, \lambda = 2a \;\Rightarrow\; k = \frac{2\pi}{\lambda} = \frac{\pi}{a}$$

$$\therefore\; E_1 = \frac{\hbar^2\pi^2}{2ma^2}$$

$$\text{For } n = 2, \lambda = a \;\Rightarrow\; k = \frac{2\pi}{\lambda} = \frac{2\pi}{a}$$

$$\therefore\; E_2 = \frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2 = \frac{4\hbar^2\pi^2}{2ma^2}$$

$$\therefore \Delta E = E_2 - E_1 = \frac{4\hbar^2\pi^2}{2ma^2} - \frac{\hbar^2\pi^2}{2ma^2} = \frac{3\hbar^2\pi^2}{2ma^2}$$

Numerically,

$$\Delta E = \frac{3\times(1.055\times10^{-34})^2\times3.1415^2}{2\times0.9\times10^{-30}\times(10^{-9})^2} = 1.831\times10^{-19}\text{ J or } \frac{1.831\times10^{-19}}{1.602\times10^{-19}} = \underline{\underline{1.143\text{eV}}}$$

(b) $h\nu = 1.831\times10^{-19} \Rightarrow \nu = \dfrac{1.831\times10^{-19}}{6.626\times10^{-34}} = 2.763\times10^{14}\text{ Hz}$

$$\therefore\; \lambda = \frac{c}{\nu} = \frac{3\times10^8}{2.763\times10^{14}} = \underline{\underline{1.09\times10^{-6}\text{ m}}}$$

$$\sin\alpha a = \sin\frac{2n\pi}{a}\cdot a = 0 \text{ and } \cos\alpha a = \cos\frac{2n\pi}{a}\cdot a = 1$$

$$\therefore \frac{1}{a}\left[\frac{2}{\alpha^3}\sin\alpha x - \frac{2x}{\alpha^2}\cos\alpha x - \frac{x^2}{\alpha}\sin\alpha x\right]_0^a = \frac{1}{a}\left(-\frac{2a}{\alpha^2}\cos\alpha a\right) = -\frac{1}{a}\frac{2a}{\left(\frac{2n\pi}{a}\right)^2} = -\frac{a^2}{2n^2\pi^2}$$

$$\therefore \langle x^2 \rangle = \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}$$

There is a misprint in this problem because u_n given by equation (3-21) is for potential well $0<x<a$ and hence $\langle x\rangle \neq 0$.

If $\langle x \rangle \neq 0$ then $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ but not $\sqrt{\langle x^2 \rangle}$ as indicated by this problem.

For this potential well, $\langle x \rangle = \frac{a}{2}$

$$\therefore \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\left(\frac{a^2}{3} - \frac{a^2}{2n^2\pi^2}\right) - \left(\frac{a}{2}\right)^2} = a\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

Now $\langle p^2 \rangle = \frac{\hbar^2\pi^2 n^2}{a^2}$, with $\langle p \rangle = 0$, $\therefore \Delta p = \sqrt{\langle p^2 \rangle} = \frac{\hbar\pi n}{a}$

$$\therefore \Delta x \Delta p = a\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}\cdot\frac{\hbar\pi n}{a} = \hbar\pi n\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

For $n = 1$, $\Delta x\Delta p = 0.568\hbar$

$n = 2$, $\Delta x\Delta p = 1.670\hbar$

$n = 3$, $\Delta x\Delta p = 2.627\hbar$

For large n, $\Delta x\Delta p \sim \hbar\pi n\sqrt{\frac{1}{12}}$ or $0.907n\hbar$