Gasiorowicz 3rd edition April 1, 2008

P5-15.

Use the commutation relation between the operators x and p to obtain the equations describing the time dependence of <x> and <p> for the Hamiltonian given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m(\omega_1{}^2 x^2 + \omega_2 x + C)$$

Solution:

$$\frac{d}{dt} < p >= \frac{i}{\hbar} < [H, p] >$$

$$= \frac{i}{\hbar} < [\frac{p^2}{2m} + \frac{1}{2}m(\omega_1{}^2 x^2 + \omega_2 x + C), p] >$$

$$= \frac{im}{2\hbar} < [(\omega_1{}^2 x^2 + \omega_2 x), p] > \qquad ([p^2, p] = 0 \text{ and } [C, p] = 0)$$

$$= \frac{im}{2\hbar}\left\{ < \omega_1{}^2 [x^2, p] > + < \omega_2 [x, p] > \right\}$$

$$= \frac{im}{2\hbar}\left\{ < \omega_1{}^2 (x[x, p] + [x, p]x > + < \omega_2 [x, p] > \right\}$$

$$= \frac{im}{2\hbar}\left\{ < \omega_1{}^2 (2i\hbar x) > + < \omega_2 (i\hbar) > \right\}$$

$$= -\frac{m}{2}\left\{ < \omega_1{}^2 (2x) > + < \omega_2 > \right\}$$

$$= -\frac{m}{2}\left\{ 2\omega_1{}^2 < x > + \omega_2 \right\} \qquad ---(1)$$

$$\frac{d}{dt} < x >= \frac{i}{\hbar} < [H, x] >$$

$$= \frac{i}{\hbar} < [\frac{p^2}{2m} + \frac{1}{2}m(\omega_1{}^2 x^2 + \omega_2 x + C), x] >$$

$$= \frac{i}{\hbar} < [\frac{p^2}{2m}, x] >$$

$$= \frac{i}{2m\hbar} < [p^2, x] >$$

$$= \frac{i}{2m\hbar} < p[p, x] + [p, x]p >$$

$$= \frac{i}{2m\hbar} < -2i\hbar p >$$

$$= \frac{< p >}{m} \qquad ----(2)$$

Differentiating (1)

$$\therefore \quad \frac{d^2}{dt^2} < p >= -\frac{m}{2}\left\{2\omega_1^{\;2} \frac{d}{dt} < x >\right\}$$

$$\Rightarrow \frac{d^2}{dt^2} < p >= -m\,\omega_1^{\;2} \frac{d}{dt} < x > \qquad \text{- - -}(3)$$

Substitute (2) into (3)

$$\frac{d^2}{dt^2} < p >= -m\,\omega_1^{\;2} \frac{< p >}{m}$$

$$\Rightarrow \frac{d^2}{dt^2} < p >= -\,\omega_1^{\;2} < p >$$

$$\Rightarrow \underline{\underline{< p > =< p(0) > e^{i\omega_1 t}}}$$

From (2) :

$$\underline{\underline{< x > = \frac{< p(0) >}{m} e^{i\omega_1 t}}}$$