Gasiorowicz 3rd edition April 22, 2008
P7-11.

A particle in a spherically symmetric potential is in a state described by the wave packet

$$\psi(x,y,z) = C\frac{(xy+yz+zx)}{r^2}e^{-\alpha r^2}$$

What is the probability that a measurement of the sphere of the angular momentum yields 0? What is the probability that it yields $6\hbar^2$? If the value of l is found to be 2, what are the relative probabilities for m=2, 1, 0. -1,-2

Solution:

$$\psi(x,y,z) = C\frac{(xy+yz+zx)}{r^2}e^{-\alpha r^2}$$

In spherical coordinates,

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$\begin{aligned}
\therefore \psi(x,y,z) &= C\frac{(xy+yz+zx)}{r^2}e^{-\alpha r^2}\\
&= C\frac{(r\sin\theta\cos\phi\cdot r\sin\theta\sin\phi + r\sin\theta\sin\phi\cdot r\cos\theta + r\cos\theta\cdot r\sin\theta\cos\phi)}{r^2}e^{-\alpha r^2}\\
&= C\frac{(r^2\sin^2\theta\cos\phi\sin\phi + r^2\sin\theta\cos\theta\sin\phi + r^2\cos\theta\sin\theta\cos\phi)}{r^2}e^{-\alpha r^2}\\
&= C(\sin^2\theta\cos\phi\sin\phi + \sin\theta\cos\theta\sin\phi + \cos\theta\sin\theta\cos\phi)e^{-\alpha r^2}\\
&= C\left[\sin^2\theta\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right)\left(\frac{e^{i\phi}-e^{-i\phi}}{2i}\right) + \sin\theta\cos\theta\left(\frac{e^{i\phi}-e^{-i\phi}}{2i}\right) + \cos\theta\sin\theta\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right)\right]e^{-\alpha r^2}\\
&= C\left[\sin^2\theta\left(\frac{e^{2i\phi}-e^{-2i\phi}}{4i}\right) + \sin\theta\cos\theta\left(\frac{e^{i\phi}-e^{-i\phi}}{2i}\right) + \cos\theta\sin\theta\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right)\right]e^{-\alpha r^2}\\
&= C\left[\frac{1}{4i}\sqrt{\frac{32\pi}{15}}\left(Y_{2,2}-Y_{2,-2}\right) + \frac{1}{2i}\sqrt{\frac{8\pi}{15}}\left(Y_{2,-1}-Y_{2,1}\right) + \frac{1}{2}\sqrt{\frac{8\pi}{15}}\left(Y_{2,-1}+Y_{2,1}\right)\right]e^{-\alpha r^2}\\
&= C\left[\frac{1}{4i}\sqrt{\frac{32\pi}{15}}\left(Y_{2,2}-Y_{2,-2}\right) + \frac{1}{2}\sqrt{\frac{8\pi}{15}}\left(\frac{1}{i}+1\right)Y_{2,-1} + \frac{1}{2}\sqrt{\frac{8\pi}{15}}\left(-\frac{1}{i}+1\right)Y_{2,-1}\right]e^{-\alpha r^2}
\end{aligned}$$

For $L^2 = 6\hbar^2, \ell = 2$

$\psi(x,y,z)$ is composed of spherical harmonics of $\ell = 2$ only. $\therefore$ the probability of yielding angular momentum 0 is 0 and the probabilied of yielding $6\hbar^2$ is 1.

$$\psi = C\left[\frac{1}{4i}\sqrt{\frac{32\pi}{15}}\left(Y_{2,2} - Y_{2,-2}\right) + \frac{1}{2}\sqrt{\frac{8\pi}{15}}\left(\frac{1}{i}+1\right)Y_{2,-1} + \frac{1}{2}\sqrt{\frac{8\pi}{15}}\left(-\frac{1}{i}+1\right)Y_{2,-1}\right]e^{-\alpha r^2}$$

$\therefore P(m = 0) = 0$

$$P(m = 2) = P(m = -2) = C^2\left[\frac{1}{4i}\sqrt{\frac{32\pi}{15}}\right]^2 = \frac{1}{16}\frac{32\pi}{15}C^2 = \frac{2\pi}{15}C^2$$

$$P(m = 1) = P(m = -1) = C^2\left[\frac{1}{2}\sqrt{\frac{8\pi}{15}}\left(\frac{1}{i}+1\right)\right]^2 = \frac{1}{4}\frac{8\pi}{15}\cdot 2C^2 = \frac{4\pi}{15}C^2$$

A^2 is determined by $\frac{2\pi}{15}C^2 + \frac{2\pi}{15}C^2 + \frac{4\pi}{15}C^2 + \frac{4\pi}{15}C^2 = 1$

$$\Rightarrow \frac{\pi}{15}C^2(12) = 1$$

$$\Rightarrow \frac{4\pi}{5}C^2 = 1$$

$$\Rightarrow C^2 = \frac{5}{4\pi}$$

$$\therefore \quad P(m = 2) = P(m = -2) = \frac{2\pi}{15}\cdot\frac{5}{4\pi} = \underline{\underline{\frac{1}{6}}}$$

$$P(m = 1) = P(m = -1) = \frac{4\pi}{15}C^2\cdot\frac{5}{4\pi} = \frac{1}{3}$$

$$P(m = 0) = \underline{\underline{0}}$$