Gasiorowicz 3rd edition April 14, 2008

P7-6.

Use lowering operator to calculate the angular dependence (without worrying about normalization) of $Y_{4m}(\theta,\phi)$ for m=3, 2, 1, 0. You are given $Y_{44}(\theta,\phi) = Ae^{4i\phi}\sin^4\theta$.

Solution:

$$Y_{44}(\theta,\phi) = Ae^{4i\phi}\sin^4\theta$$

$$L_- = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\therefore Y_{43}(\theta,\phi) = L_-\, Y_{44}(\theta,\phi)$$

$$= B'\,e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)e^{4i\phi}\sin^4\theta$$

$$= B'\,e^{-i\phi}\left(-4\sin^3\theta\cos\theta - 4\cot\theta\sin^4\theta\right)e^{4i\phi}$$

$$= B'\,e^{-i\phi}\left(-4\sin^3\theta\cos\theta - 4\sin^3\theta\cos\theta\right)e^{4i\phi}$$

$$= B\sin^3\theta\cos\theta\, e^{3i\phi} \qquad (B = -8B')$$

$$Y_{42}(\theta,\phi) = L_-\, Y_{43}(\theta,\phi)$$

$$= C\,e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\sin^3\theta\cos\theta\, e^{3i\phi}$$

$$= C\,e^{-i\phi}\left(\sin^4\theta - 3\sin^2\theta\cos^2\theta - 3\cot\theta\sin^3\theta\cos\theta\right)e^{3i\phi}$$

$$= C\,e^{-i\phi}\left(\sin^4\theta - 3\sin^2\theta\cos^2\theta - 3\cos^2\theta\sin^2\theta\right)e^{3i\phi}$$

$$= C\,e^{2i\phi}\left(\sin^4\theta - 6\sin^2\theta\cos^2\theta\right)$$

$$= C\,e^{2i\phi}\left[\sin^4\theta - 6\sin^2\theta(1-\sin^2\theta)\right]$$

$$= C\,e^{2i\phi}\left[7\sin^4\theta - 6\sin^2\theta\right]$$

$$Y_{41}(\theta,\phi) = L_-\, Y_{42}(\theta,\phi)$$

$$= D'\,e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)e^{2i\phi}\left[7\sin^4\theta - 6\sin^2\theta\right]$$

$$= D'\,e^{-i\phi}\left[-28\sin^3\theta\cos\theta + 12\sin\theta\cos\theta - 2\cot\theta\left(7\sin^4\theta - 6\sin^2\theta\right)\right]e^{2i\phi}$$

$$= D'\,e^{-i\phi}\left[-28\sin^3\theta\cos\theta + 12\sin\theta\cos\theta - 14\cos\theta\sin^3\theta + 12\sin\theta\cos\theta\right]e^{2i\phi}$$

$$= D'e^{i\phi}\left(-42\sin^3\theta\cos\theta + 24\sin\theta\cos\theta\right)$$

$$= De^{i\phi}\left(4\sin\theta\cos\theta - 7\sin^3\theta\cos\theta\right) \qquad (D = 6D')$$

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\begin{aligned}
Y_{40}(\theta,\phi) &= L_- \; Y_{41}(\theta,\phi) \\
&= E\, e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\, e^{i\phi}\left(4\sin\theta\cos\theta - 7\sin^3\theta\cos\theta\right) \\
&= E\, e^{-i\phi}\left[\left(-4\cos^2\theta + 4\sin^2\theta + 21\sin^2\theta\cos^2\theta - 7\sin^4\theta\right) - \left(4\sin\theta\cos\theta - 7\sin^3\theta\cos\theta\right)\cot\theta\right]\, e^{i\phi} \\
&= E\, e^{-i\phi}\left[-4\cos^2\theta + 4\sin^2\theta + 21\sin^2\theta\cos^2\theta - 7\sin^4\theta - 4\cos^2\theta + 7\sin^2\theta\cos^2\theta\right] e^{i\phi} \\
&= E\left(-8\cos^2\theta + 4\sin^2\theta + 28\sin^2\theta\cos^2\theta - 7\sin^4\theta\right) \\
&= E\left[-8(1-\sin^2\theta) + 4\sin^2\theta + 28\sin^2\theta(1-\sin^2\theta) - 7\sin^4\theta\right] \\
&= E\left[-8 + 8\sin^2\theta + 4\sin^2\theta + 28\sin^2\theta - 28\sin^4\theta - 7\sin^4\theta\right] \\
&= E\left[-8 + 40\sin^2\theta - 35\sin^4\theta\right]
\end{aligned}
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