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P8-7

An electron in the Coulomb field of a proton is in a state described by the wave function

$$\frac{1}{6}\left[4\psi_{100}(\bar{r})+3\psi_{211}(\bar{r})-\psi_{210}(\bar{r})+\sqrt{10}\psi_{21-1}(\bar{r})\right]$$

(a) What is the expectation value of the energy?

(b) What is the expectation value of L?

(c) What is the expectation value of L_z?

Solution:

$$\frac{1}{6}\left[4\psi_{100}(\bar{r})+3\psi_{211}(\bar{r})-\psi_{210}(\bar{r})+\sqrt{10}\psi_{21-1}(\bar{r})\right]$$

(a)

$$E_n=-\frac{1}{2}\mu c^2\frac{(Z\alpha^2)}{n^2}$$

$$\text{With } Z=1, E_n=-\frac{1}{2}\mu c^2\alpha^2\frac{1}{n^2}$$

$$E \text{ for } \psi_{100} = E_1=-\frac{1}{2}\mu c^2\alpha^2$$

$$E \text{ for } \psi_{211} = E_2=-\frac{1}{2}\mu c^2\alpha^2\times\frac{1}{4}$$

$$E \text{ for } \psi_{210} = E_2=-\frac{1}{2}\mu c^2\alpha^2\times\frac{1}{4}$$

$$E \text{ for } \psi_{21-1} = E_2=-\frac{1}{2}\mu c^2\alpha^2\times\frac{1}{4}$$

$$\therefore <E>=\left(-\frac{1}{2}\mu c^2\alpha^2\right)\left(\frac{4}{6}\right)^2+\left(-\frac{1}{2}\mu c^2\alpha^2\times\frac{1}{4}\right)\left(\frac{3}{6}\right)^2+\left(-\frac{1}{2}\mu c^2\alpha^2\times\frac{1}{4}\right)\left(\frac{1}{6}\right)^2++\left(-\frac{1}{2}\mu c^2\alpha^2\times\frac{1}{4}\right)\left(\frac{\sqrt{10}}{6}\right)^2$$

$$=\left(-\frac{1}{2}\mu c^2\alpha^2\right)\times\frac{16}{36}+\left(-\frac{1}{2}\mu c^2\alpha^2\times\frac{1}{4}\right)\times\frac{20}{36}$$

$$=\left(-\frac{1}{2}\mu c^2\alpha^2\right)\times\frac{21}{36}$$

$$=-\frac{7}{24}\mu c^2\alpha^2$$

(b) L^2 for $\psi_{100}=0(0+1)\hbar=0$

L^2 for $\psi_{211}=1(1+1)\hbar=2\hbar^2$

L^2 for $\psi_{210}=1(1+1)\hbar=2\hbar^2$

L^2 for $\psi_{21-1}=1(1+1)\hbar=2\hbar^2$

$$\therefore \langle L^2 \rangle = (0)\left(\frac{4}{6}\right)^2 + (2\hbar^2)\left(\frac{3}{6}\right)^2 + (2\hbar^2)\left(\frac{1}{6}\right)^2 + (2\hbar^2)\left(\frac{\sqrt{10}}{6}\right)^2$$

$$= (2\hbar^2)\times\frac{20}{36}$$

$$= \frac{40}{36}\hbar^2$$

$$= \underline{\underline{\frac{10\hbar}{9}}}$$

(c) L_z for $\psi_{100} = 0\hbar$

L_z for $\psi_{211} = 1\hbar$

L_z for $\psi_{210} = 0\hbar$

L_z for $\psi_{21-1} = -\hbar$

$$\therefore \langle m \rangle = (0)\left(\frac{4}{6}\right)^2 + (1\hbar)\left(\frac{3}{6}\right)^2 + (0)\left(\frac{1}{6}\right)^2 + (-\hbar)\left(\frac{\sqrt{10}}{6}\right)^2$$

$$= \frac{9}{36}\hbar - \frac{10}{36}\hbar$$

$$= \underline{\underline{\frac{1}{36}\hbar}}$$