Gasiorowicz 3rd edition April 25, 2008
P8-11

An electron in the Coulomb field of a proton is in a state described by the wave function

$$\psi(\vec{r}) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{3/2} e^{-\alpha^2 r^2/2}$$

Write out an expression for the probability that it will be found in the ground state of the hydrogen atom.

Solution:

Ground state radial wave function = $\psi_{\text{ground}} = \dfrac{2}{\sqrt{2\pi}}\left(\dfrac{1}{a_0}\right)^{\frac{3}{2}} e^{-r/a_0}$ $\qquad (Z=1)$

$$\psi(\vec{r}) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{3/2} e^{-\alpha^2 r^2/2}$$

$\therefore$ Probability for the electron to be found in the ground state

$$= |\langle \psi_{\text{ground}} | \psi(\vec{r}) \rangle|^2$$

$$= \left(\int_0^\infty \left(\frac{\alpha}{\sqrt{\pi}}\right)^{3/2} e^{-\alpha^2 r^2/2} \cdot \frac{2}{\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}} e^{-r/a_0} \cdot 4\pi r^2 dr\right)^2$$

$$= \frac{32\alpha^3}{a_0{}^3\sqrt{\pi}}\left(\int_0^\infty e^{-\alpha^2 r^2/2} \cdot e^{-r/a_0} \cdot r^2 dr\right)^2$$

$$= \underline{\underline{\frac{32\alpha^3}{a_0{}^3\sqrt{\pi}}\left(\int_0^\infty e^{-(\alpha^2 r^2/2 + r/a_0)} r^2 dr\right)^2}}$$