Gasiorowicz 3rd edition April 25, 2008
P8-5.

Work out the Schrodinger equation in polar coordinates ρ, ϕ, with x= ρcos ϕ, y= ρsin ϕ, for a potential that depends only on ρ. If the solution of the equation ψ(ρ, ϕ) is written as R(ρ)Φ(ϕ), what is the equation obeyed by Φ(ϕ)? What is the equation for R(ρ)?

Solution:

$$x = \rho\cos\phi \Rightarrow dx = \cos\phi\, d\rho - \rho\sin\phi d\phi \quad ---(1)$$
$$y = \rho\sin\phi \Rightarrow dy = \sin\phi\, d\rho + \rho\cos\phi d\phi \quad ---(2)$$
$$(1)\cos\phi + (2)\sin\phi \Rightarrow d\rho = \cos\phi dx + \sin\phi dy \quad ----(3)$$
$$(2)\cos\phi - (1)\sin\phi \Rightarrow \rho d\phi = -\sin\phi dx + \cos\phi dy \quad ----(4)$$

$$\frac{\partial}{\partial x} = \frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} = \cos\phi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\phi\frac{\partial}{\partial\phi}$$
$$\frac{\partial^2}{\partial x^2} = \left(\cos\phi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\phi\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\phi\frac{\partial}{\partial\phi}\right)$$
$$= \cos^2\phi\frac{\partial^2}{\partial\rho^2} + \frac{\sin^2\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2} - \frac{2\sin\phi\cos\phi}{\rho}\frac{\partial^2}{\partial\rho\partial\phi} \quad ----(5)$$
$$\frac{\partial}{\partial y} = \frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi} = \sin\phi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\phi\frac{\partial}{\partial\phi}$$
$$\frac{\partial^2}{\partial y^2} = \left(\sin\phi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\phi\frac{\partial}{\partial\phi}\right)\left(\sin\phi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\phi\frac{\partial}{\partial\phi}\right)$$
$$= \sin^2\phi\frac{\partial^2}{\partial\rho^2} + \frac{\cos^2\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{2\sin\phi\cos\phi}{\rho}\frac{\partial^2}{\partial\rho\partial\phi} \quad ----(6)$$
$$(5)+(6)\Rightarrow$$
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \cos^2\phi\frac{\partial^2}{\partial\rho^2} + \frac{\sin^2\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2} - \frac{2\sin\phi\cos\phi}{\rho}\frac{\partial^2}{\partial\rho\partial\phi}$$
$$+\sin^2\phi\frac{\partial^2}{\partial\rho^2} + \frac{\cos^2\phi}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{2\sin\phi\cos\phi}{\rho}\frac{\partial^2}{\partial\rho\partial\phi}$$
$$= \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}$$

Schrödinger equation :

$$-\frac{\hbar^2}{2M}\nabla^2\Psi + V\Psi = E\Psi \Rightarrow \underline{\underline{-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)\Psi + V(\rho)\Psi = E\Psi}}$$

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)\Psi+V(\rho)\Psi=E\Psi$$

$$\Psi(\rho,\phi)=R(\rho)\Phi(\phi)$$

$$\therefore -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)R(\rho)\Phi(\phi)+V(\rho)R(\rho)\Phi(\phi)=ER(\rho)\Phi(\phi)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)R(\rho)\Phi(\phi)-\frac{2MV(\rho)}{\hbar^2}R(\rho)\Phi(\phi)=-\frac{2ME}{\hbar^2}R(\rho)\Phi(\phi)$$

$$\Rightarrow \Phi(\phi)\frac{\partial^2}{\partial\rho^2}R(\rho)+\frac{R(\rho)}{\rho^2}\frac{\partial^2}{\partial\phi^2}\Phi(\phi)-\frac{2MV(\rho)}{\hbar^2}R(\rho)\Phi(\phi)=-\frac{2ME}{\hbar^2}R(\rho)\Phi(\phi)$$

$$\Rightarrow \frac{1}{R(\rho)}\frac{\partial^2}{\partial\rho^2}R(\rho)+\frac{1}{\Phi(\phi)\rho^2}\frac{\partial^2}{\partial\phi^2}\Phi(\phi)-\frac{2MV(\rho)}{\hbar^2}=-\frac{2ME}{\hbar^2}$$

$$\Rightarrow \frac{\rho^2}{R(\rho)}\frac{\partial^2}{\partial\rho^2}R(\rho)-\frac{2M\rho^2V(\rho)}{\hbar^2}+\frac{2M\rho^2E}{\hbar^2}= -\frac{1}{\Phi(\phi)}\frac{\partial^2}{\partial\phi^2}\Phi(\phi) \qquad ----(7)$$

Left hand side depends on ρ and right hand side depends on ϕ, this can be possible only if they equal to a constant, say, m^2 :

$$-\frac{1}{\Phi(\phi)}\frac{\partial^2}{\partial\phi^2}\Phi(\phi)=m^2 \Rightarrow \underline{\underline{\frac{\partial^2}{\partial\phi^2}\Phi(\phi)=-m^2\Phi(\phi)}} \qquad ----(8)$$

Solution of this equation is $\underline{\underline{\Phi(\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi}}}$, $m=0, \pm 1, \pm 2, ...$

Left hand side of equation (7) :

$$\frac{\rho^2}{R(\rho)}\frac{\partial^2}{\partial\rho^2}R(\rho)-\frac{2M\rho^2V(\rho)}{\hbar^2}+\frac{2M\rho^2E}{\hbar^2}=m^2 \Rightarrow \underline{\underline{\frac{\partial^2}{\partial\rho^2}R(\rho)-\frac{m^2}{\rho^2}R(\rho)+\frac{2M}{\hbar^2}(E-V(\rho)R(\rho)=0}}$$