Ex. 2-13.

(a) $T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. $T^{\dagger} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

$$\therefore TT^{\dagger} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$T^{\dagger}T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{pmatrix}.$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\therefore$ T is unitary.

(b). To determine Eigenvalues and eigenvectors:

$$\det|T - I\lambda| = 0 \Rightarrow \det\begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\cos\theta - \lambda)^2 + \sin^2\theta = 0.$$

$$\Rightarrow \cos^2\theta - 2\cos\theta\lambda + \lambda^2 + \sin^2\theta = 0.$$

$$\Rightarrow \lambda^2 - 2\cos\theta\lambda + 1 = 0.$$

$$\Rightarrow \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$\Rightarrow \lambda = \frac{2\cos\theta \pm 2i\sqrt{1-\cos^2\theta}}{2}.$$

$$= \underline{\cos\theta \pm i\sin\theta} \quad \text{or } \underline{e^{\pm i\theta}}$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (\cos\theta \pm i\sin\theta)\begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\Rightarrow \begin{cases} x\cos\theta + y\sin\theta = (\cos\theta \pm i\sin\theta)x \\ -x\sin\theta + y\cos\theta = (\cos\theta \pm i\sin\theta)y. \end{cases}$$

$$\Rightarrow \begin{cases} y\sin\theta = \pm i\sin\theta\, x \\ -x\sin\theta = \pm i\sin\theta\, y. \end{cases}$$

$$\Rightarrow \begin{cases} y = \pm ix \\ -x = \pm iy. \Rightarrow iy = \mp x \Rightarrow -y = \mp ix \Rightarrow y = \pm ix \end{cases}$$ Dependent!

Let $x = 1$. $y = \pm i$

$\therefore$ The Eigenvectors are

$$\begin{pmatrix} 1 \\ i \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

After normalization.

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ with Eigenvalue $\cos\theta + i\sin\theta$.

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ with Eigenvalue $\cos\theta - i\sin\theta$.