Ex 2-22.

(a)
$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad A^{\dagger} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = A$$

So A is Hermitian.

To determine Eigenvalues of A (Expecting all real):

$$\det(A - I\lambda) = 0 \Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda \begin{vmatrix} -1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & -1-\lambda \\ 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(-1-\lambda)(-\lambda) + (1+\lambda) = 0.$$

$$\Rightarrow -\lambda^2(1+\lambda) + (1+\lambda) = 0.$$

$$\Rightarrow (1-\lambda^2)(1+\lambda) = 0$$

$$\Rightarrow (1-\lambda)(1+\lambda)(1+\lambda) = 0$$

$$\Rightarrow \lambda = 1, -1, \text{ or } -1.$$

$\therefore$ The Eigenvalues are $1, -1$ and -1. (all real!)

To determine Eigenvectors:

For $\lambda = 1$:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} z = x. \\ -y = y. \\ x = z. \end{cases} \Rightarrow \begin{cases} x = z \\ y = 0. \end{cases}$$

Choose $x = z = 1/\sqrt{2}$

$\therefore$ The normalized Eigenvector is $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda = -1$.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} \cancel{x = -z}\ z = -x \\ -y = -y \\ x = -z. \end{cases}$$

$\therefore$ The Eigenvectors are $\begin{pmatrix} \alpha \\ \beta \\ -\alpha \end{pmatrix}$.

Let us choose $\alpha = 0$ and $\beta = 1$.

$\therefore$ The second Eigenvector is

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The third one has to be choosen carefully, otherwise it will not orthogonal with $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

$\therefore$ The choice of α and β has to satisfy requirement:

$$(\alpha \quad \beta \quad -\alpha)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \Rightarrow \beta = 0.$$

$\therefore$ Let us ~~use~~ choose $\alpha = \frac{1}{\sqrt{2}}$.

The Third Eigenvector is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

$\therefore$ The Three Eigenvectors are

$$|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Note the choice of $|2\rangle$ and $|3\rangle$ is NOT unique.
To show orthonormality of these 3 Eigenvectors:

$$\langle 1|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} = \frac{1}{2}\cdot 2 = 1.$$

$$\langle 2|2\rangle = \begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = 1.$$

$$\langle 3|3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \frac{1}{2}\cdot 2 = 1.$$

$$\langle 1|2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = 0.$$

$$\langle 2|3\rangle = \begin{pmatrix}0 & 1 & 0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} = 0.$$

$$\langle 1|3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$
$$= \frac{1}{2}(1-1) = 0.$$

To show completeness:

$$|1\rangle\langle 1| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0 & 1\\0 & 0 & 0\\1 & 0 & 1\end{pmatrix}.$$

$$|2\rangle\langle 2| = \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}.$$

$$|3\rangle\langle 3| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0 & -1\\0 & 0 & 0\\-1 & 0 & 1\end{pmatrix}.$$

$$\therefore |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| = \cancel{\tfrac{1}{2}}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = I.$$

(b). $|1\rangle\langle 1| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

$|2\rangle\langle 2| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$|3\rangle\langle 3| = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$