

Ex. 2-8 $a(2, -3, 0) + b(0, 0, 1) + c(2i, i, -i) = 0$
 $\Rightarrow \begin{cases} 2a + 0b + 2ic = 0 \\ -3a + 0b + ic = 0 \\ 0a + b - ic = 0 \end{cases}$

(a). $a(2, -3, 0) + b(0, 0, 1) + c(2i, i, -i) = 0$

$\Rightarrow \begin{cases} 2a + 0b + 2ic = 0 \\ -3a + 0b + ic = 0 \\ 0a + b - ic = 0 \end{cases}$

$\begin{vmatrix} 2 & 0 & 2i \\ -3 & 0 & i \\ 0 & 1 & -i \end{vmatrix} = 2(-i) + (2i)(-3) = -8i$

$\therefore a = b = c = 0$

$(2, -3, 0), (0, 0, 1) \& (2i, i, -i)$ are independent.

(b). $a(0, 4, 0) + b(i, -3i, i) + c(2, 0, 1) = 0$

$\Rightarrow \begin{cases} 0a + ib + 2c = 0 \\ 4a - 3ib + 0c = 0 \\ 0a + ib + 1c = 0 \end{cases}$

$\begin{vmatrix} 0 & i & 2 \\ 4 & -3i & 0 \\ 0 & i & 1 \end{vmatrix} = (-4)(i - 2i) = 4i \neq 0$

$\therefore a = b = c = 0$

$(0, 4, 0), (i, -3i, i) \& (2, 0, 1)$ are independent.

(c). $a(2, -3, 0) + b(0, 0, 1) + c(2i, i, -i) = 0$

$\Rightarrow \begin{cases} 2a + 0b + 2ic = 0 \\ -3a + 0b + ic = 0 \\ 0a + b - ic = 0 \end{cases}$

$\begin{vmatrix} 2 & 0 & 2i \\ -3 & 0 & i \\ 0 & 1 & -i \end{vmatrix} = (-1)(2i + 6i) = -8i \neq 0$

$\therefore (2, -3, 0), (i, -3i, i), (0, 0, 1) \& (2i, i, -i)$ are independent.

$$(d) a(i, 1, 2) + b(3, i, -1) + c(-i, 3i + 5i) = 0$$

$$(a) \Rightarrow \begin{cases} ia + 3b - ic = 0 \\ ia + ib + 3ic = 0 \\ 2a + \cancel{-b} + 5ic = 0 \end{cases} \quad T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{vmatrix} i & 3 & -i \\ 1 & i & 3i \\ 2 & -1 & 5i \end{vmatrix} = i(-5 + 3i) - 3(5i - 6i) - i(-1 - 2i) \\ = -5i - 3 - 15i + 18i + i - 2 \\ = -2i - 5 \neq 0.$$

$\therefore (i, 1, 2), (3, i, -1) \& (-i, 3i + 5i)$ are linearly independent.

$\therefore T$ is unitary

(b). To determine eigenvalues and eigenvectors:

$$\det |T - I\lambda| = 0 \Rightarrow \det \begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\cos\theta - \lambda)^2 + \sin^2\theta = 0.$$

$$\Rightarrow \cos^2\theta - 2\cos\theta\lambda + \lambda^2 + \sin^2\theta = 0.$$

$$\Rightarrow \lambda^2 - 2\cos\theta\lambda + 1 = 0.$$

$$\Rightarrow \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$\Rightarrow \lambda = \frac{2\cos\theta \pm 2\sqrt{1 - \cos^2\theta}}{2}$$

$$= \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$