Ex 4.16

(a) If we measure energy, ~~three~~ ~~thus~~ there will be 3 possible outcomes:

$$E_1 = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2. \qquad P(E_1) = \frac{4}{4+3+4} = \frac{4}{11}.$$

$$E_2 = \frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2. \qquad P(E_2) = \frac{3}{4+3+4} = \frac{3}{11}$$

$$E_3 = \frac{\hbar^2}{2m}\left(\frac{3\pi}{a}\right)^2. \qquad P(E_3) = \frac{4}{4+3+4} = \frac{4}{11}.$$

$$\langle E\rangle = E_1 P(E_1) + E_2 P(E_2) + E_3 P(E_3).$$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 \cdot \frac{4}{11} + \frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2 \cdot \frac{3}{11} + \frac{\hbar^2}{2m}\left(\frac{3\pi}{a}\right)^2 \cdot \frac{4}{11}$$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 \left[\frac{4}{11} + 4\cdot\frac{3}{11} + 9\cdot\frac{4}{11}\right].$$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 \cdot \frac{52}{11}$$

$$= \frac{26\hbar^2}{11}\left(\frac{\pi}{a}\right)^2 //$$

(b) $\psi(x,t)$

$$= \frac{2}{\sqrt{7a}}\sin\left(\frac{\pi x}{a}\right)e^{-i\frac{\hbar}{2m}\left(\frac{\pi}{a}\right)^2 t} + \sqrt{\frac{3}{7a}}\sin$$

$$= \frac{2}{\sqrt{7a}}\sin\left(\frac{\pi x}{a}\right)e^{-i\frac{E_1}{\hbar}t} + \sqrt{\frac{3}{7a}}\sin\frac{2\pi x}{a}e^{-i\frac{E_2}{\hbar}t}$$

$$+ \frac{2}{\sqrt{7a}}\sin\frac{3\pi x}{a}e^{-i\frac{E_3}{\hbar}t}.$$

$$P(E_3) = \langle E_3 | \psi(x,t) \rangle$$

$$= \frac{3}{4+3+4} = \frac{3}{11} //$$

(c). Let us first normalize the wavefunction:

$$\psi(x,t=0) = \sqrt{\frac{4}{11}} \cdot \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a} + \sqrt{\frac{3}{11}} \cdot \sqrt{\frac{2}{a}} \sin\frac{2\pi x}{a} + \sqrt{\frac{4}{11}} \cdot \sqrt{\frac{2}{a}} \sin\frac{3\pi x}{a}$$

$$= \sqrt{\frac{8}{11a}} \sin\frac{\pi x}{a} + \sqrt{\frac{6}{11a}} \sin\frac{2\pi x}{a} + \sqrt{\frac{8}{11a}} \sin\frac{3\pi x}{a}.$$

$$\psi(x,t) = \sqrt{\frac{8}{11a}} e^{-\frac{iE_1}{\hbar}t} \sin\frac{\pi x}{a} + \sqrt{\frac{6}{11a}} e^{-\frac{iE_2}{\hbar}t} \sin\frac{2\pi x}{a} + \sqrt{\frac{8}{11a}} e^{-\frac{iE_3}{\hbar}t} \sin\frac{3\pi x}{a}.$$

$$\therefore \rho(x,t) = \psi^*(x,t)\,\psi(x,t).$$

$$= \frac{8}{11a}\sin^2\frac{\pi x}{a} + \frac{6}{11a}\sin^2\frac{2\pi x}{a} + \frac{8}{11a}\sin^2\frac{3\pi x}{a}.$$

$$+ \frac{\sqrt{48}}{11a}\left(e^{+i\frac{(E_2-E_1)}{\hbar}t} \sin\frac{\pi x}{a}\sin\frac{2\pi x}{a} + e^{-i\frac{(E_2-E_1)}{\hbar}t} \sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\right).$$

$$+ \frac{\sqrt{48}}{11a}\left(e^{i\frac{(E_2-E_2)}{\hbar}t} + e^{-i\frac{(E_3-E_2)}{\hbar}t}\right)\sin\frac{2\pi x}{a}\sin\frac{3\pi x}{a}$$

$$+ \frac{\sqrt{64}}{11a}\left(e^{i\frac{(E_3-E_1)}{\hbar}t} + e^{-i\frac{(E_3-E_1)}{\hbar}t}\right)\sin\frac{\pi x}{a}\sin\frac{3\pi x}{a}.$$

$$= \frac{8}{11a}\sin^2\frac{\pi x}{a} + \frac{6}{11a}\sin^2\frac{2\pi x}{a} + \frac{8}{11a}\sin^2\frac{3\pi x}{a}$$

$$+ \frac{2\sqrt{48}}{11a}\cos\frac{(E_2-E_1)}{\hbar}t \sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}$$

$$+ \frac{2\sqrt{48}}{11a}\cos\frac{(E_3-E_2)}{\hbar}t \sin\frac{2\pi x}{a}\sin\frac{3\pi x}{a}$$

$$+ \frac{2\times 8}{11a}\cos\frac{(E_3-E_1)}{\hbar}t \sin\frac{\pi x}{a}\sin\frac{3\pi x}{a}. //$$

$$j(x,t) = \frac{-i\hbar}{2m}\left(\psi \frac{d\psi^*}{dx} - \psi^* \frac{d}{dx}\psi\right)$$

$$\psi \frac{d}{dx}\psi^* = \left[\sqrt{\frac{8}{11a}} e^{-\frac{iE_1}{\hbar}t} \sin\frac{\pi x}{a} + \sqrt{\frac{6}{11a}} e^{-\frac{iE_2}{\hbar}t} \sin\frac{2\pi x}{a} + \sqrt{\frac{8}{11a}} e^{-\frac{iE_3}{\hbar}t} \sin\frac{3\pi x}{a}\right]$$

$$\times \left[\sqrt{\frac{8}{11a}} \cdot \frac{\pi}{a} e^{\frac{iE_1}{\hbar}t} \cos\frac{\pi x}{a} + \sqrt{\frac{6}{11a}} \cdot \frac{2\pi}{a} e^{\frac{iE_2}{\hbar}t} \cos\frac{2\pi x}{a} + \sqrt{\frac{8}{11a}} \cdot \frac{3\pi}{a} e^{\frac{iE_3}{\hbar}t} \cos\frac{3\pi x}{a}\right]$$

$$= \frac{8}{11a} \cdot \frac{\pi}{a} \sin\frac{\pi x}{a}\cos\frac{\pi x}{a} + \frac{6}{11a} \cdot \frac{2\pi}{a} \sin\frac{2\pi x}{a}\cos\frac{2\pi x}{a} + \frac{8}{11a} \cdot \frac{3\pi}{a} \sin\frac{3\pi x}{a}\cos\frac{3\pi x}{a}$$

$$+ \sqrt{\frac{48}{11a}} e^{\frac{i(E_2-E_1)}{\hbar}t} + \sqrt{\frac{48}{11a}} \cdot \frac{2\pi}{a} e^{\frac{iE_2-E_1}{\hbar}t} \sin\frac{\pi x}{a}\cos\frac{2\pi x}{a} + \frac{8}{11a} \cdot \frac{3\pi}{a} e^{\frac{iE_3-E_2}{\hbar}t} \sin\frac{\pi x}{a}\cos\frac{3\pi x}{a}$$

$$+ \sqrt{\frac{48}{11a}} \frac{\pi}{a} e^{\frac{i(E_1-E_2)}{\hbar}t} \sin\frac{2\pi x}{a}\cos\frac{\pi x}{a} + \sqrt{\frac{48}{11a}} \cdot \frac{3\pi}{a} e^{\frac{i(E_3-E_2)}{\hbar}t} \sin\frac{2\pi x}{a}\cos\frac{3\pi x}{a}$$

$$+ \frac{8}{11a} \cdot \frac{\pi}{a} e^{\frac{i(E_1-E_3)}{\hbar}t} \sin\frac{3\pi x}{a}\cos\frac{\pi x}{a} + \sqrt{\frac{48}{11a}} \cdot \frac{2\pi}{a} e^{\frac{i(E_2-E_3)}{\hbar}t} \cos\frac{2\pi x}{a}\sin\frac{3\pi x}{a}$$

$$\psi\frac{d\psi^*}{dx} - \psi^*\frac{d\psi}{dx} = 2i\left[\sqrt{\frac{48}{11a}} \cdot \frac{2\pi}{a} \sin\left(\frac{E_2-E_1}{\hbar}\right)t \sin\frac{\pi x}{a}\cos\frac{2\pi x}{a} + \frac{8}{11a} \cdot \frac{3\pi}{a} \sin\left(\frac{E_3-E_1}{\hbar}\right)t \sin\frac{\pi x}{a}\cos\frac{3\pi x}{a}\right.$$

$$+ \sqrt{\frac{48}{11a}} \cdot \frac{\pi}{a} \sin\left(\frac{E_1-E_2}{\hbar}\right)t \sin\frac{2\pi x}{a}\cos\frac{\pi x}{a} + \sqrt{\frac{48}{11a}} \cdot \frac{3\pi}{a} \sin\left(\frac{E_3-E_2}{\hbar}\right)t \sin\frac{2\pi x}{a}\cos\frac{3\pi x}{a}$$

$$\left. + \frac{8}{11a} \cdot \frac{\pi}{a} \sin\left(\frac{E_1-E_3}{\hbar}\right)t \sin\frac{3\pi x}{a}\cos\frac{\pi x}{a} + \sqrt{\frac{48}{11a}} \cdot \frac{2\pi}{a} \sin\left(\frac{E_2-E_3}{\hbar}\right)t \sin\frac{3\pi x}{a}\cos\frac{2\pi x}{a}\right]$$

$$j(x,t) = -\frac{\hbar}{m}\left[\sqrt{\frac{48}{11a}} \cdot \frac{2\pi}{a} \sin\left(\frac{E_2-E_1}{\hbar}\right)t \sin\frac{\pi x}{a}\cos\frac{2\pi x}{a} + \frac{8}{11a} \cdot \frac{3\pi}{a} \sin\left(\frac{E_3-E_1}{\hbar}\right)t \sin\frac{\pi x}{a}\cos\frac{3\pi x}{a}\right.$$

$$+ \sqrt{\frac{48}{11a}} \cdot \frac{\pi}{a} \sin\left(\frac{E_1-E_2}{\hbar}\right)t \sin\frac{2\pi x}{a}\cos\frac{\pi x}{a} + \sqrt{\frac{48}{11a}} \cdot \frac{3\pi}{a} \sin\left(\frac{E_3-E_2}{\hbar}\right)t \sin\frac{2\pi x}{a}\cos\frac{3\pi x}{a}$$

$$\left. + \frac{8}{11a} \cdot \frac{\pi}{a} \sin\left(\frac{E_1-E_3}{\hbar}\right)t \sin\frac{3\pi x}{a}\cos\frac{\pi x}{a} + \sqrt{\frac{48}{11a}} \cdot \frac{2\pi}{a} \sin\left(\frac{E_2-E_3}{\hbar}\right)t \sin\frac{3\pi x}{a}\cos\frac{2\pi x}{a}\right]. \;//$$

$$\frac{dj(x,t)}{dx} = -\frac{\hbar}{2m}\left[\sqrt{\frac{48}{11a}}\cdot\frac{2\pi}{a}\cdot\frac{\pi}{a}\sin\left(\frac{E_2-E_1}{\hbar}\right)t\cos\frac{\pi x}{a}\cos\frac{2\pi x}{a} - \sqrt{\frac{48}{11a}}\cdot\left(\frac{2\pi}{a}\right)^2\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\sin\left(\frac{E_2-E_1}{\hbar}\right)t\right.$$

$$+\frac{8}{11a}\cdot\frac{3\pi}{a}\cdot\frac{\pi}{a}\sin\left(\frac{E_3-E_1}{\hbar}\right)t\cos\frac{\pi x}{a}\cos\frac{3\pi x}{a} - \frac{8}{11a}\cdot\left(\frac{3\pi}{a}\right)^2\sin\left(\frac{E_3-E_1}{\hbar}\right)t\sin\frac{\pi x}{a}\sin\frac{3\pi x}{a}.$$

$$+\sqrt{\frac{48}{11a}}\cdot\frac{\pi}{a}\cdot\frac{2\pi}{a}\sin\left(\frac{E_1-E_2}{\hbar}\right)t\cos\frac{2\pi x}{a}\cos\frac{\pi x}{a} - \sqrt{\frac{48}{11a}}\cdot\left(\frac{\pi}{a}\right)^2\sin\left(\frac{E_1-E_2}{\hbar}\right)t\sin\frac{2\pi x}{a}\sin\frac{\pi x}{a}.$$

$$+\sqrt{\frac{48}{11a}}\cdot\frac{3\pi}{a}\cdot\frac{2\pi}{a}\sin\left(\frac{E_3-E_2}{\hbar}\right)t\cos\frac{2\pi x}{a}\cos\frac{3\pi x}{a} - \sqrt{\frac{48}{11a}}\left(\frac{3\pi}{a}\right)^2\sin\left(\frac{E_3-E_2}{\hbar}\right)t\sin\frac{2\pi x}{a}\sin\frac{3\pi x}{a},$$

$$+\frac{8}{11a}\cdot\frac{\pi}{a}\cdot\frac{3\pi}{a}\sin\left(\frac{E_1-E_3}{\hbar}\right)t\cos\frac{3\pi x}{a}\cos\frac{\pi x}{a} - \frac{8}{11a}\cdot\left(\frac{\pi}{a}\right)^2\sin\left(\frac{E_1-E_3}{\hbar}\right)t\sin\frac{3\pi x}{a}\sin\frac{\pi x}{a}$$

$$\left.+\sqrt{\frac{48}{11a}}\cdot\frac{2\pi}{a}\cdot\frac{3\pi}{a}\sin\left(\frac{E_2-E_3}{\hbar}\right)t\cos\frac{3\pi x}{a}\cos\frac{\pi x}{a} - \sqrt{\frac{48}{11a}}\cdot\left(\frac{2\pi}{a}\right)^2\sin\left(\frac{E_2-E_3}{\hbar}\right)t\sin\frac{3\pi x}{a}\sin\frac{2\pi x}{a}\right]$$

$$= \frac{\hbar}{m}\left[\sqrt{\frac{48}{11a}}\left(\frac{2\pi}{a}\right)^2\sin\left(\frac{E_2-E_1}{\hbar}\right)t\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a} + \frac{8}{11a}\left(\frac{3\pi}{a}\right)^2\sin\left(\frac{E_3-E_1}{\hbar}\right)t\sin\frac{\pi x}{a}\sin\frac{3\pi x}{a}\right.$$

$$+\sqrt{\frac{48}{11a}}\left(\frac{\pi}{a}\right)^2\sin\left(\frac{E_1-E_2}{\hbar}\right)t\sin\frac{2\pi x}{a}\sin\frac{\pi x}{a} + \sqrt{\frac{48}{11a}}\left(\frac{3\pi}{a}\right)^2\sin\left(\frac{E_3-E_2}{\hbar}\right)t\sin\frac{2\pi x}{a}\sin\frac{3\pi x}{a}$$

$$\left.+\frac{8}{11a}\left(\frac{\pi}{a}\right)^2\sin\left(\frac{E_1-E_3}{\hbar}\right)t\sin\frac{3\pi x}{a}\sin\frac{\pi x}{a} + \sqrt{\frac{48}{11a}}\left(\frac{2\pi}{a}\right)^2\sin\left(\frac{E_2-E_3}{\hbar}\right)t\sin\frac{3\pi x}{a}\sin\frac{2\pi x}{a}\right].$$

$$= \frac{\hbar}{m}\left[\sqrt{\frac{48}{11a}}\left(\left(\frac{2\pi}{a}\right)^2-\left(\frac{\pi}{a}\right)^2\right)\sin\left(\frac{E_2-E_1}{\hbar}\right)t\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\right.$$

$$+\sqrt{\frac{48}{11a}}\left(\left(\frac{3\pi}{a}\right)^2-\left(\frac{2\pi}{a}\right)^2\right)\sin\left(\frac{E_3-E_2}{\hbar}\right)t\sin\frac{2\pi x}{a}\sin\frac{3\pi x}{a}.$$

$$\left.+\frac{8}{11a}\left(\left(\frac{3\pi}{a}\right)^2-\left(\frac{\pi}{a}\right)^2\right)\sin\left(\frac{E_3-E_1}{\hbar}\right)t\sin\frac{\pi x}{a}\sin\frac{3\pi x}{a}\right].$$

$$= \frac{2}{\hbar}\left[\sqrt{\frac{48}{11a}}\left(E_2-E_1\right)\sin\left(\frac{E_2-E_1}{\hbar}\right)t\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\right.$$

$$+\sqrt{\frac{48}{11a}}\left(E_3-E_2\right)\sin\left(\frac{E_3-E_2}{\hbar}\right)t\sin\frac{2\pi x}{a}\sin\frac{3\pi x}{a}$$

$$\left.+\frac{8}{11a}\left(E_3-E_1\right)\sin\left(\frac{E_3-E_1}{\hbar}\right)t\sin\frac{\pi x}{a}\sin\frac{3\pi x}{a}\right].$$

$$\frac{d\rho}{dt} = -2\cdot\frac{\sqrt{48}}{11a}\cdot\frac{(E_2-E_1)}{\hbar}\sin\frac{(E_2-E_1)}{\hbar}t\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}$$

$$-2\cdot\frac{\sqrt{48}}{11a}\frac{(E_3-E_2)}{\hbar}\sin\left(\frac{E_3-E_2}{\hbar}\right)t\sin\frac{2\pi x}{a}\sin\frac{3\pi x}{a}$$

$$-2\cdot\frac{8}{11a}\frac{(E_3-E_1)}{\hbar}\sin\left(\frac{E_3-E_1}{\hbar}\right)t\sin\frac{\pi x}{a}\sin\frac{3\pi x}{a}.$$

$$=-\frac{2}{\hbar}\left[\frac{\sqrt{48}}{11a}(E_2-E_1)\sin\frac{(E_2-E_1)}{\hbar}t\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\right.$$

$$+\frac{\sqrt{48}}{11a}(E_3-E_2)\sin\frac{(E_3-E_2)}{\hbar}t\sin\frac{2\pi x}{a}\sin\frac{3\pi x}{a}$$

$$\left.+\frac{8}{11a}(E_3-E_1)\sin\frac{(E_3-E_1)}{\hbar}t\sin\frac{\pi x}{a}\sin\frac{3\pi x}{a}\right].$$

$$\therefore \frac{d\rho}{dt}+\frac{dj}{dx}=0.$$