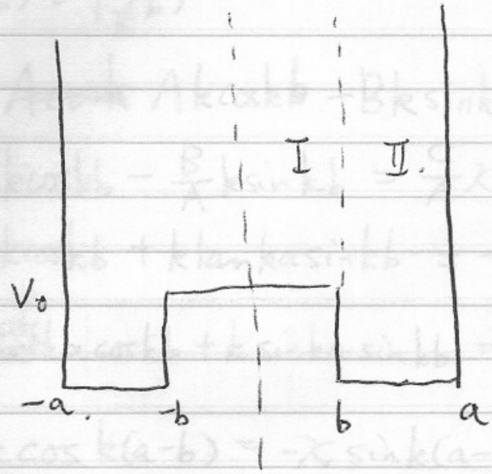
~~Ex 3.4.~~ Ex 4.4

~~(a). $\hat{H}\psi_0(x)$~~

(a).



Since $V(x)$ is symmetric. $\therefore$ The solution must be either odd or even. We need only to consider $x>0$ for our solution.

Case 1. $\psi(x)$ is even.

$$\psi_{II} = A\sin kx + B\cos kx \qquad \left(k = \frac{\sqrt{2mE}}{\hbar}\right).$$

$$\psi_{I} = C\cosh \varkappa x \qquad \left(\varkappa = \frac{\sqrt{2m(V_0-E)}}{\hbar}\right).$$

$$\psi_{II}(a)=0 \Rightarrow A\sin ka + B\cos ka = 0.$$

$$\Rightarrow \frac{B}{A} = -\tan ka. \qquad \text{—(1).}$$

$$\psi_I(b) = \psi_{II}(b) \Rightarrow A\sin kb + B\cos kb = C\cosh\varkappa b.$$

$$\Rightarrow \sin kb - \tan ka \cos kb = \frac{C}{A}\cosh\varkappa b.$$

$$\Rightarrow \sin kb\cos ka - \sin ka\cos kb = \frac{C}{A}\cosh\varkappa b\cos ka$$

$$\Rightarrow \sin k(b-a) = \frac{C}{A}\cosh\varkappa b\cos ka.$$

$$\Rightarrow \frac{C}{A} = -\frac{\sin k(a-b)}{\cosh\varkappa b\cos ka}. \qquad \text{—(2).}$$

$$\psi_I'(b) = \psi_{II}'(b)$$

$$\Rightarrow \cancel{A\cos k}\; Ak\cos kb - Bk\sin kb = C\kappa \overset{\sinh}{\cancel{\cosh}}\, \kappa b.$$

$$\Rightarrow k\cos kb - \frac{B}{A}k\sin kb = \frac{C}{A}\kappa \sinh \kappa b$$

$$\Rightarrow k\cos kb + k\tan ka \sin kb = -\kappa \frac{\sin k(a-b)}{\cosh \kappa b \cos ka} \sinh \kappa b.$$

$$\Rightarrow k\overset{\cos}{\cancel{\sin}}ka\cos kb + k\sin ka \sin kb = -\kappa \frac{\sin k(a-b)}{\cosh \kappa b} \sin kb.$$

$$\Rightarrow k\cos k(a-b) = -\kappa \sin k(a-b) \tanh \kappa b.$$

$$\Rightarrow \boxed{\tan k(a-b) \tanh \kappa b = -\frac{k}{\kappa}.} \Rightarrow E_1$$

Case 2. $\psi(x)$ is odd.

$$\psi_{II} = A\sin kx + B\cos kx$$

$$\psi_I = C\sinh \kappa x.$$

$$\psi_{II}(0) \Rightarrow \frac{B}{A} = -\tan ka. \quad \text{— (1)}$$

$$\psi_I(b) = \psi_{II}(b) \Rightarrow \frac{C}{A} = -\frac{\sin k(a-b)}{\sinh \kappa b \cos ka} \quad \text{— (2)}.$$

$$\psi_I'(b) = \psi_{II}'(b) \Rightarrow Ak\cos kb - Bk\sin kb = C\kappa \cosh \kappa b.$$

$$\Rightarrow k\cos kb + k\tan ka \sin kb = -\kappa \frac{\sin k(a-b)}{\sinh \kappa b \cos ka} \cosh \kappa b.$$

$$\Rightarrow k\cos k(a-b) = -\kappa \sin k(a-b) \coth \kappa b.$$

$$\Rightarrow \boxed{\tan k(a-b) \coth \kappa b = -\frac{k}{\kappa}} \Rightarrow E_2.$$

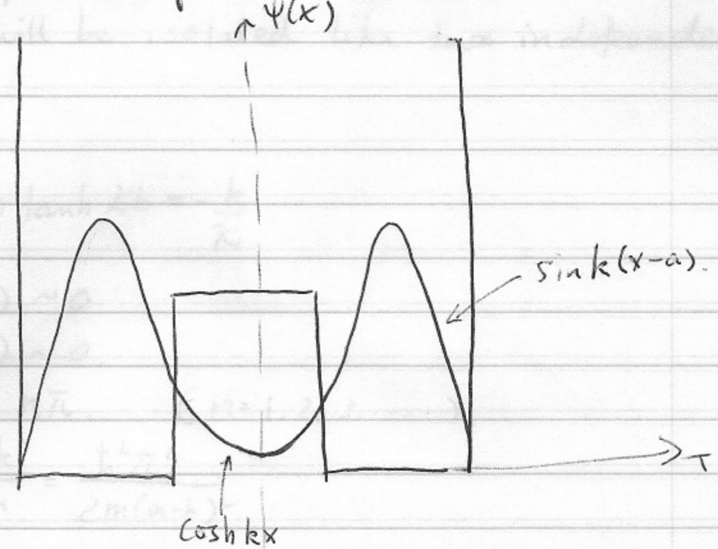
Notice that with $\cancel{A}\ \frac{B}{A} = -\tan ka.$

$$\psi_{II} = A(\sin kx - \tan ka \cos kx)$$

$$= \frac{-A}{\cos ka} \sin(a-x) \text{ or } \frac{A}{\cos ka} \sin(x-a).$$

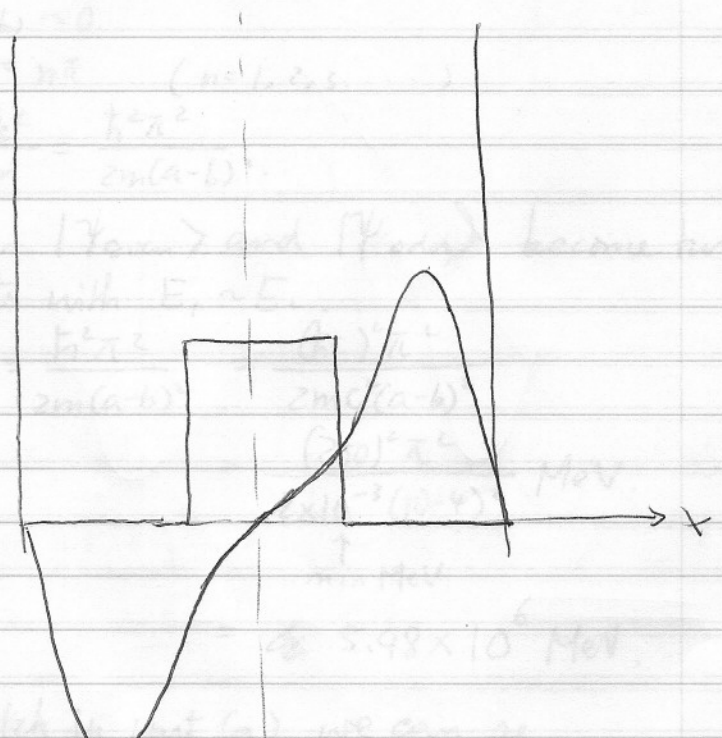
Ground state is the evenfunction. $(E = E_1)$

$|\psi_{even}\rangle$:



1st excited state is odd. $(E = E_2)$

$|\psi_{\cancel{even}\ odd}\rangle$.



(b) We consider approximate case $\kappa b \gg 1 \Rightarrow \kappa \gg k$ and the two sides will be isolated like two independent wells:

For $|\psi_{even}\rangle$.

$$\tan k(a-b) \tanh \kappa b = -\frac{k}{\kappa}.$$

$$\Rightarrow \tan k(a-b) \sim 0.$$

$$\Rightarrow \sin k(a-b) \sim 0.$$

$$\Rightarrow k(a-b) = n\pi. \quad (n=1,2,3,\dots).$$

$$\therefore E_1 = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2\pi^2}{2m(a-b)^2}.$$

For $|\psi_{odd}\rangle$.

$$\tan k(a-b) \coth \kappa b = -\frac{k}{\kappa}.$$

$$\Rightarrow \tan k(a-b) \sim 0.$$

$$\Rightarrow k(a-b) = n\pi \quad (n=1,2,3\dots)$$

$$E_2 = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2\pi^2}{2m(a-b)^2}.$$

$\therefore$ When $\kappa b = 1$, $|\psi_{even}\rangle$ and $|\psi_{odd}\rangle$ become two degenerate states with $E_1 \sim E_2$.

$$E_1 \sim E_2 = \frac{\hbar^2\pi^2}{2m(a-b)^2} = \frac{(\hbar c)^2\pi^2}{2mc^2(a-b)^2}$$

$$= \frac{(200)^2\pi^2}{2\times10^{-3}(10-4)^2} \text{ MeV}.$$

$\uparrow$ min MeV

$$= 5.48\times10^6 \text{ MeV}.$$

(b). From the sketch in part (a), we can se

$$|\psi_{left}\rangle \sim \frac{1}{\sqrt{2}}\left(|\psi_{even}\rangle - |\psi_{odd}\rangle\right)$$

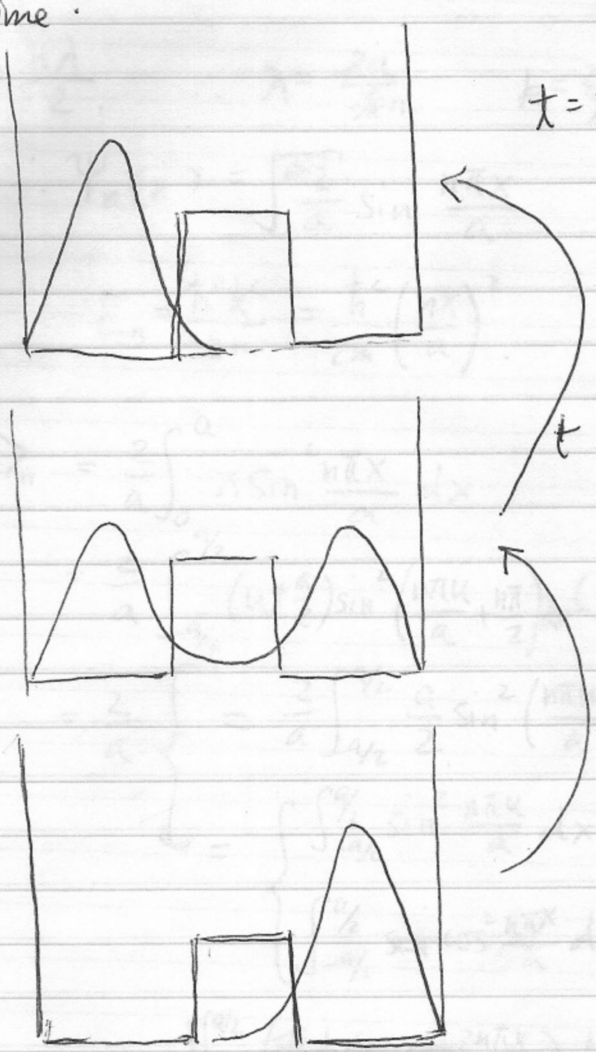
and $|\psi_{right}\rangle \sim \frac{1}{\sqrt{2}}\left(|\psi_{even}\rangle + |\psi_{odd}\rangle\right)$

Evolution with time:

$t=0$: $t=T$.

$t=\frac{1}{4}T$. $t=\frac{3}{4}T$

$t=\frac{1}{2}T$



The particles "osillates" between the left and the ~~wi~~ right with a frequency ~~dete~~ determined by the difference of E_1 and E_2.

$$\omega \sim \frac{E_2-E_1}{\hbar}.$$