Ex. 4.7

(a). $a = \dfrac{n\lambda}{2}$. $\quad \therefore \lambda = \dfrac{2a}{n}$. $\quad k = \dfrac{2\pi}{\lambda} = 2\pi \cdot \dfrac{n}{2a} = \dfrac{n\pi}{a}$.

$$\therefore \psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \qquad n = 1, 2, 3 \ldots\ldots$$

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2.$$

(b).
$$\langle x \rangle_n = \frac{2}{a}\int_0^a x \sin^2 \frac{n\pi x}{a}\, dx$$

$$= \frac{2}{a}\int_{-a/2}^{a/2} \left(u + \frac{a}{2}\right) \sin^2\left(\frac{n\pi u}{a} + \frac{n\pi}{2}\right) dx \quad \left(u = x - \frac{a}{2}\right).$$

$$\cancel{= \frac{2}{a}} = \frac{2}{a}\int_{-a/2}^{a/2} \frac{a}{2} \sin^2\left(\frac{n\pi u}{a} + \frac{n\pi}{2}\right) dx$$

$$= \begin{cases} \displaystyle\int_{-a/2}^{a/2} \sin^2 \frac{n\pi u}{a}\, dx & n \text{ Even} \\ \displaystyle\int_{-a/2}^{a/2} \cancel{\sin}\cos^2 \frac{n\pi x}{a}\, dx & n \text{ odd.} \end{cases}$$

$$= \begin{cases} \displaystyle\int_{-a/2}^{a/2} \frac{1}{2}\left(1 - \cos \frac{2n\pi x}{a}\right) dx & n \text{ Even.} \\ \displaystyle\int_{-a/2}^{a/2} \frac{1}{2}\left(1 + \cos \frac{2n\pi x}{a}\right) dx & n \text{ odd.} \end{cases}$$

$$= \underline{\underline{\frac{a}{2}}} \quad \leftarrow \text{obvious result!}$$

Similary, we can show

$$\langle p \rangle_n = 0. \,//$$ (also obvious result).

Now, $\langle x^2 \rangle_n$.

Integration ~~tabl~~ table:

$$\int x^2 \sin^2 ax \, dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax - \frac{x\cos 2ax}{4a^2}.$$

$$\therefore \langle x^2 \rangle_n = \frac{2}{a}\int_0^a x^2 \sin^2 \frac{n\pi x}{a}\,dx = \frac{2}{a}\left\{\left[\frac{x^3}{6}\right]_0^a - \left[\left(\frac{x^2}{4\alpha} - \frac{1}{8\alpha^3}\right)\sin 2\alpha x\right]_0^a - \left[\frac{x\cos 2\alpha x}{4\alpha^2}\right]_0^a\right\} \quad \left(\alpha = \frac{n\pi}{a}\right).$$

$$= \frac{2}{a}\left\{\frac{a^3}{6} - 0 - \frac{a\cos 2n\pi}{4\left(\frac{n\pi}{a}\right)^2}\right\}.$$

$$= \frac{2}{a}\left\{\frac{a^3}{6} - \frac{a^3}{4n^2\pi^2}\right\}.$$

$$= a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) \;//.$$

$$\langle p^2 \rangle_n = \frac{2}{a}\int_0^a \sin\frac{n\pi x}{a}\cdot\left(-\hbar^2\frac{d^2}{dx^2}\right)\sin\frac{n\pi x}{a}\,dx$$

$$= +\frac{2\hbar^2}{a^2}\cdot\left(\frac{n\pi}{a}\right)^2\int_0^a \sin^2\frac{n\pi x}{a}\,dx$$

$$= 2\left(\frac{n\hbar\pi}{a^2}\right)^2\cdot\int_0^a \frac{1}{2}\left(1 - \cos\frac{2n\pi x}{a}\right)dx.$$

$$= 2\cdot\left(\frac{n\hbar\pi}{a^2}\right)^2\cdot\frac{a}{2}.$$

$$= \left(\frac{n\hbar\pi}{a}\right)^2.$$

(obvious as $k = \frac{n\pi}{a}$).

$$\cancel{\Delta x \Delta p = \sqrt{\langle \Delta x \rangle^2 -}}$$

$$\Delta x \Delta p = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}\ \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$= \sqrt{a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) - \left(\frac{a}{2}\right)^2} \cdot \sqrt{\left(\frac{n\hbar\pi}{a}\right)^2 - 0}.$$

$$= a \cdot \sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}} \cdot \frac{n\hbar\pi}{a}.$$

$$= \sqrt{\frac{\cancel{6}n^2\pi^2 - 6}{12n^2\pi^2}} \cdot n\hbar\pi.$$

$$= \hbar\sqrt{\frac{n^2\pi^2 - 6}{12}} \quad //$$