Extra HW problem.

Construct new basis:

$$|b_1\rangle = \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix} \quad |b_2\rangle = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \quad \text{and } |b_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(This correspond in rotating the original $\hat{x}$ and $\hat{y}$ together with an angle of θ).

If is obvious that $\langle b_3|b_1\rangle = 0$ and $\langle b_3|b_2\rangle = 0$.

$$\langle b_1|b_2\rangle = (\cos\theta \;\; -\sin\theta \;\; 0)\begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}.$$

$$= \cos\theta\sin\theta - \sin\theta\cos\theta = 0.$$

$$\langle b_1|b_1\rangle = \cos^2\theta + \sin^2\theta = 1$$

$$\langle b_2|b_2\rangle = \sin^2\theta + \cos^2\theta = 1.$$

$\therefore$ $|b_1\rangle$, $|b_2\rangle$ and $|b_3\rangle$ form an orthonormal basis.

Change of basis matrix:

$$T = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore T^{+} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$TT^{+} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

similarly, we can prove $T^{+}T = I$.

$\therefore$ T is unitary.