

Extra HW problem.

Construct new basis:

$$|b_1\rangle = \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix} \quad |b_2\rangle = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \quad \text{and} \quad |b_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(This corresponds to rotating the original $\hat{x}$ and $\hat{y}$ together with an angle of θ).

It is obvious that $\langle b_3 | b_1 \rangle = 0$ and $\langle b_3 | b_2 \rangle = 0$.

$$\langle b_1 | b_2 \rangle = (\cos\theta \quad -\sin\theta \quad 0) \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

$$= \cos\theta \sin\theta - \sin\theta \cos\theta = 0.$$

$$\langle b_1 | b_1 \rangle = \cos^2\theta + \sin^2\theta = 1$$

$$\langle b_2 | b_2 \rangle = \sin^2\theta + \cos^2\theta = 1.$$

$\therefore |b_1\rangle, |b_2\rangle$ and $|b_3\rangle$ form an orthonormal basis.

Change of basis matrix:

$$T = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ex. 1. $T^+ = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$TT^+ = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2 \theta + \sin^2 \theta & -\cos \theta \sin \theta + \sin \theta \cos \theta & 0 \\ -\sin \theta \cos \theta + \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly, we can prove $T^+T = I$.

$\therefore T$ is unitary.

(c). $a(2, -3, 0) + b(0, 0, 1) + c(2i, i, -i) = 0$

$$\Rightarrow \begin{cases} 2a + 2ic = 0 \\ -3a + b - ic = 0 \\ 0a + b - i = 0 \end{cases}$$

$$\begin{vmatrix} 2 & 0 & 2i \\ -3 & 1 & -i \\ 0 & 1 & -i \end{vmatrix} = (-1)(2i + 6i) = -8i \neq 0.$$

$\therefore (2, -3, 0), (0, 0, 1), (2i, i, -i)$ are independent.