Hermitian and unitary operator

An operator is Hermitian if it is self-adjoint:

$$A^+ = A$$

Or equivalently: $\langle \psi|A|\phi\rangle = (\langle\phi|A|\psi\rangle)^*$ and so $\langle A\rangle = \langle \psi|A|\psi\rangle$ is real.

An operator is skew-Hermitian if $B^+ = -B$ and $\langle B\rangle = \langle \psi|B|\psi\rangle$ is imaginary.

In quantum mechanics, the expectation of any physical quantity has to be real and hence *an operator corresponds to a physical observable must be Hermitian*. For example, momentum operator and Hamiltonian are Hermitian.

An operator is Unitary if its inverse equal to its adjoints:

$$U^{-1} = U^+ \quad \text{or} \quad UU^+ = U^+U = I$$

In quantum mechanics, unitary operator is used for change of basis.

Commutators

Operators do not commute.

Commutator: $[A, B] = AB-BA$

Anti-commutator: $\{A, B\} = AB+BA$

<u>Algebra of commutators:</u>

1. Antisymmetric: $[A, B] = -[B, A]$
2. Bilinear: $[\alpha A+\beta B+\gamma C+\dots , X] = \alpha [A,X]+ \beta[B,X]+ \gamma[C,X]+\dots$

   $[X, \alpha A+\beta B+\gamma C+\dots] = \alpha[X, A]+ \beta[X, B]+ \gamma[X, C]+\dots$
3. Adjoint: $[A, B]^+ = [B^+, A^+]$
4. Distributive: $[A, BC] = [A,B]C+B[A,C]$

   $[BC, A] = [B,A]C+B[C,A]$
5. $$[A,B^n]=\sum_{j=0}^{n-1} B^j[A,B]A^{n-j-1}$$

   $$[A^n,B]=\sum_{j=0}^{n-1} A^{n-j-1}[A,B]B^j$$
6. Jacobi identity: $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$
7. If A and B are Hermitian, $[A, B]$ is skew-Hermitian and $\{A, B\}$ is Hermitian.
8. Note that $[A, B]=0$ and $[B, C]=0$ does *NOT* imply $[B, C]=0$.

Function of operators

If f(A) is a "function" of A. We can Taylor expand f(A) in a power series of A:

$$f(A) = \sum_{n=0}^{\infty} a_n A^n$$

$$\therefore [f(A)]^+ = \sum_{n=0}^{\infty} a_n^{\,*} (A^+)^n = f^*(A^+)$$

For example,

$$e^{aA} = \sum_{n=0}^{\infty} \frac{a^n A^n}{n!} = I + aA + \frac{1}{2} a^2 A^2 + \frac{1}{6} a^3 A^3 + \cdots$$

$$(e^A)^+ = e^{A^+} \quad \text{and} \quad (e^{iA})^+ = e^{-iA^+}$$

Commutators involving function of operators :

1. [A, f(A)] = 0
2. $e^A e^B \neq e^{A+B}$ but $e^A e^B = e^{A+B} e^{[A,B]/2}$
3. $e^{-A} B e^A = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots$

Commutators and uncertainty relation

Define operator

$\delta A = A - \langle A \rangle$ where $\langle A \rangle = \langle \psi | A | \psi \rangle$ and $A2 = \langle \psi | A2 | \psi \rangle$

Let $\delta A|\psi\rangle = |\chi\rangle$ and $\delta B|\psi\rangle = |\phi\rangle$. Note that if A and B are Hermitian, so are δA and δB.

$\delta A = \delta A^+ \Rightarrow \langle \delta A^2 \rangle = \langle \delta A \delta A \rangle = \langle \psi | \delta A \delta A | \psi \rangle = \langle \psi | \delta A^+ \delta A | \psi \rangle = \langle \chi | \chi \rangle \Rightarrow \langle \delta A^2 \rangle = \langle \chi | \chi \rangle$

Similarly, $\langle \delta B^2 \rangle = \langle \phi | \phi \rangle$ and $\langle \delta A \delta B \rangle = \langle \chi | \phi \rangle$

Schwarz inequality: $\langle \chi | \chi \rangle \langle \phi | \phi \rangle \geq |\langle \chi | \phi \rangle|^2 \Rightarrow \langle \delta A^2 \rangle \langle \delta B^2 \rangle \geq \langle \delta A \delta B \rangle^2$

Now,

$$\left\langle \delta A \delta B \right\rangle = \left\langle \tfrac{1}{2}(\delta A \delta B - \delta B \delta A) + \tfrac{1}{2}(\delta A \delta B + \delta B \delta A) \right\rangle$$

$$= \tfrac{1}{2}\left\langle [\delta A, \delta B] \right\rangle + \tfrac{1}{2}\left\langle \{\delta A, \delta B\} \right\rangle$$

$$= \underbrace{\tfrac{1}{2}\left\langle [A, B] \right\rangle}_{\text{Imaginary}} + \underbrace{\tfrac{1}{2}\left\langle \{\delta A, \delta B\} \right\rangle}_{\text{Real}}$$

$$\therefore \left|\left\langle \delta A \delta B \right\rangle\right|^2 = \tfrac{1}{4}\left|\left\langle [A, B] \right\rangle\right|^2 + \tfrac{1}{2}\left\langle \{\delta A, \delta B\} \right\rangle^2$$

$$\left\langle \delta A^2 \right\rangle \left\langle \delta B^2 \right\rangle \geq \left\langle \delta A \delta B \right\rangle^2 \Rightarrow \left\langle \delta A^2 \right\rangle \left\langle \delta B^2 \right\rangle \geq \tfrac{1}{4}\left|\left\langle [A, B] \right\rangle\right|^2 + \tfrac{1}{2}\left\langle \{\delta A, \delta B\} \right\rangle^2$$

$$\Rightarrow \quad \left\langle \delta A^2 \right\rangle \left\langle \delta B^2 \right\rangle \geq \tfrac{1}{4}\left|\left\langle [A, B] \right\rangle\right|^2$$

Define $\Delta A = \sqrt{\left\langle \delta A^2 \right\rangle} = \sqrt{\left\langle \left(A - \langle A \rangle\right)^2 \right\rangle} = \sqrt{\left\langle A^2 \right\rangle - \left\langle A \right\rangle^2}$ and $\Delta B = \sqrt{\left\langle \delta B^2 \right\rangle}$

$$\therefore \Delta A^2 \Delta B^2 \geq \tfrac{1}{4}\left|\left\langle [A, B] \right\rangle\right|^2 \quad \Rightarrow \quad \boxed{\Delta A \Delta B \geq \tfrac{1}{2}\left|\left\langle [A, B] \right\rangle\right|}$$