Use of Ladder operators

Let the ladder operators a and a^+ for J_1, J_2, and J be J_{1-}, J_{2-}, J_- and J_{1+}, J_{2+}, J_+:

$$| j, m_J >= \sum_{m_1', m_2'} | m_1' m_2' >< m_1' m_2' | j m_J >$$

$$\Rightarrow < m_1 m_2 | J_\pm | j, m_J >= \sum_{m_1', m_2'} < m_1 m_2 | J_\pm | m_1' m_2' >< m_1' m_2' | j m_J >$$

$$\Rightarrow < m_1 m_2 | J_\pm | j, m_J >= \sum_{m_1', m_2'} < m_1 m_2 | J_{1\pm} + J_{2\pm} | m_1' m_2' >< m_1' m_2' | j m_J >$$

$$\Rightarrow < m_1 m_2 | j, m_J \pm 1 > \sqrt{j(j+1) - m_J (m_J \pm 1)}\hbar$$

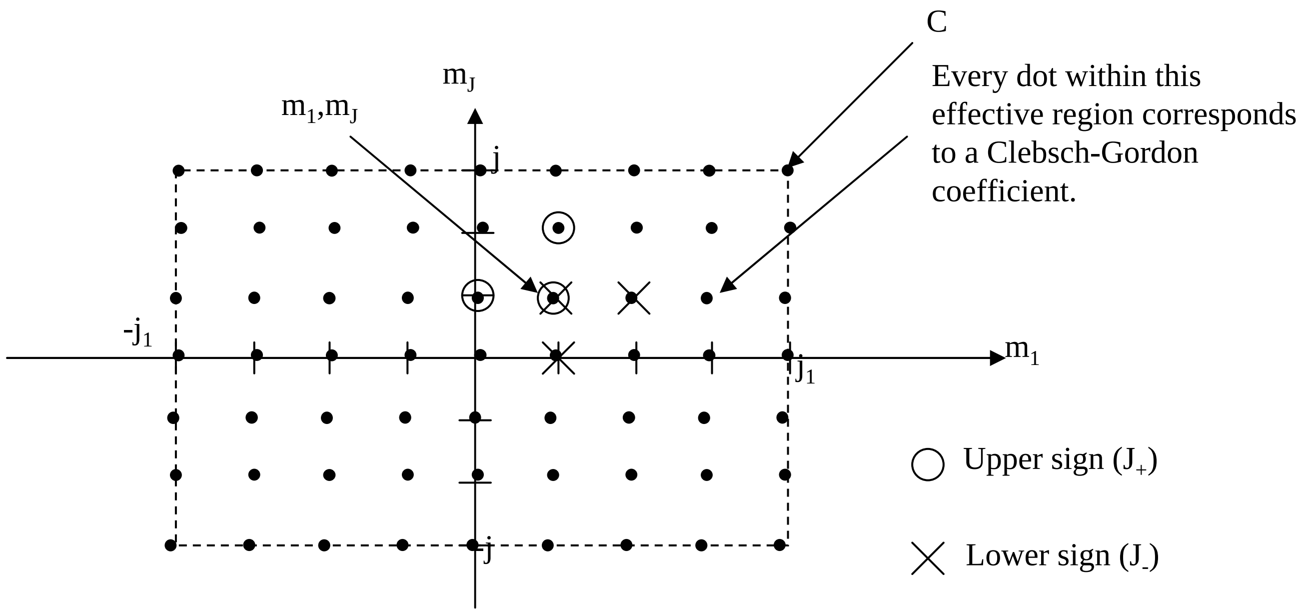
$$= \sum_{m_1', m_2'} [< m_1 m_2 | m_1' \pm 1, m_2' >< m_1' m_2' | j m_J > \sqrt{j_1 (j_1 + 1) - m_1'(m_1' \pm 1)}\hbar$$

$$+ < m_1 m_2 | m_1', m_2' \pm 1 >< m_1' m_2' | j m_J > \sqrt{j_2 (j_2 + 1) - m_2'(m_2' \pm 1)}\hbar]$$

$$\Rightarrow < m_1 m_2 | j, m_J \pm 1 > \sqrt{j(j+1) - m_J (m_J \pm 1)}$$

$$= [< m_1 \mp 1, m_2 | j m_J > \sqrt{j_1 (j_1 + 1) - m_1 (m_1 \mp 1)} + < m_1, m_2 \mp 1 | j m_J > \sqrt{j_2 (j_2 + 1) - m_2 (m_2 \mp 1)}]$$

Strategy in using Ladder operators



Let the ladder operators a and a^+ for J_1, J_2, and J be J_{1-}, J_{2-}, J_- and J_{1+}, J_{2+}, J_+:

1. Start from the top right corner, $m_1=j_1$, $m_j=j$, and $m_2=j-j_1$. Let the Clebsch-Gordon coefficient be C. Using the step down operator (lower sign) J_-, we can figure our all the Clebsch-Gordon coefficient at the right edge of the rectangle.
2. After this, we can use the step up operator (upper sign) J_+ to fill up the rectangle column by column, starting from the right edge.
3. At the end, we will get all the Clebsch-Gorden coefficients in terms of C. C is to be determined by normalization, or the unitary of the of the transformation matrix.

Spin ½ + Spin ½

Let us consider the case of adding two ½ spins, i.e. j_1=1/2 and j_2=1/2. Dimension of the product space is 2x2=4.

We will use the notation to represent these *non-interacting* basis vectors:

|↑ ↑> ≡ | m_1=1/2 , m_2=1/2>
|↑ ↓> ≡ | m_1=1/2 , m_2= -1/2>
| ↓↑> ≡ | m_1=-1/2 , m_2=1/2>
| ↓↓> ≡ | m_1= -1/2 , m_2= -1/2>

In interacting representation, j=0 or 1. If j=0, m_J=0. If j=1, m_J=-1, 0, or 1. So there are also 4 basis vectors, consistent with the fact that it is a 4 dimensional space. We will use |j, m_J> to represent these *interacting* basis vectors:

|0, 0> ≡ | j=0 , m_J=1>
|1. 1> ≡ | j=1 , m_J=1>
|1, 0> ≡ | j=1 , m_J=0>
|1, -1> ≡ | j=1 , m_J=-1>

These two sets of basis vectors can be expressed as linear combination of each other. To do this, we have to calculate the Clebsch-Gordon coefficients.

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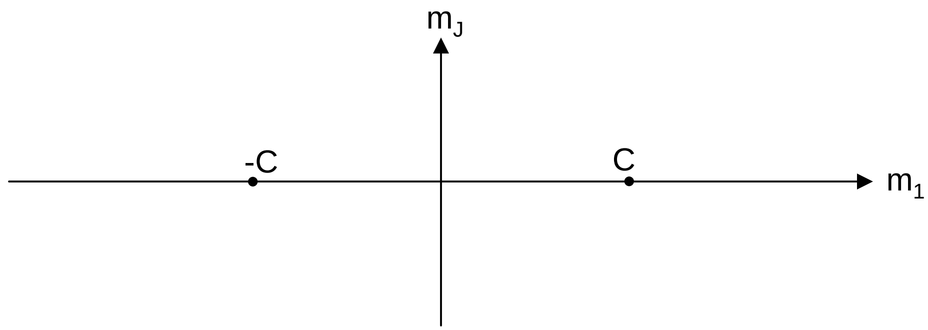
Clebsch-Gordon coefficients for Spin ½ + Spin ½

Case 1. First consider the case of j=0 (so m_J must be 0).



$$< m_1 m_2 \mid j, m_J \pm 1 > \sqrt{j(j+1) - m_J(m_J \pm 1)}$$

$$= [< m_1 \mp 1, m_2 \mid j\, m_J > \sqrt{j_1(j_1+1) - m_1(m_1 \mp 1)} + < m_1, m_2 \mp 1 \mid j\, m_J > \sqrt{j_2(j_2+1) - m_2(m_2 \mp 1)}]$$

$$\text{Upper sign with } m_1 = \frac{1}{2} \;\Rightarrow\; < \frac{1}{2}\frac{1}{2} \mid 0,1 > \sqrt{0-0}$$

$$= < -\frac{1}{2}, \frac{1}{2} \mid 0\,0 > \sqrt{\frac{1}{2} \times \frac{3}{2} - \frac{1}{2}(-\frac{1}{2})} + < \frac{1}{2}, -\frac{1}{2} \mid 00 > \sqrt{\frac{1}{2} \times \frac{3}{2} - \frac{1}{2}(-\frac{1}{2})}$$

$$\Rightarrow < -\frac{1}{2}, \frac{1}{2} \mid 0\,0 > = - < \frac{1}{2}, -\frac{1}{2} \mid 0\,0 > = -C$$

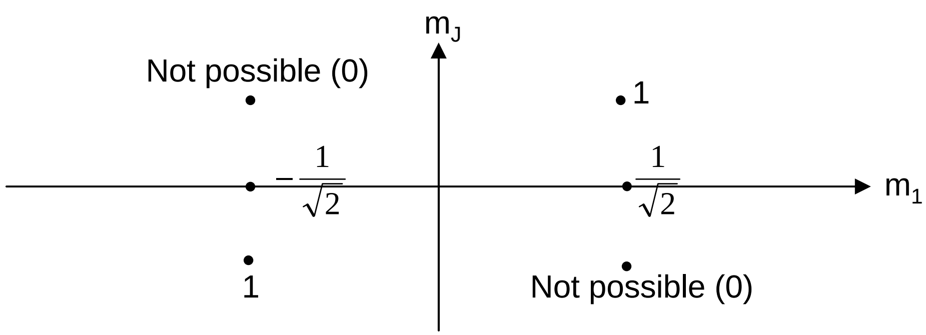
$$\text{Hence, } \mid j, m_J > = \mid 0, 0 > = C \mid \frac{1}{2}, -\frac{1}{2} > - C \mid -\frac{1}{2}, \frac{1}{2} >$$

$$< 0,0 \mid 0,0 > = 1 \;\Rightarrow\; C^2 + C^2 = 1 \Rightarrow C = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \mid 0, 0 > = \frac{1}{\sqrt{2}} \mid \frac{1}{2}, -\frac{1}{2} > - \frac{1}{\sqrt{2}} \mid -\frac{1}{2}, \frac{1}{2} > = \frac{1}{\sqrt{2}} \mid \uparrow, \downarrow > - \frac{1}{\sqrt{2}} \mid \downarrow, \uparrow > \qquad \text{(singlet)}$$

Clebsch-Gordon coefficients for Spin ½ + Spin ½

Case 2. Now the more complicated case of j=1 (so m_J can be -1, 0, or 1).



$$|1,1> = |\uparrow\uparrow>$$

$$|1,0> = \frac{1}{\sqrt{2}}|\uparrow\downarrow> + \frac{1}{\sqrt{2}}|\downarrow\uparrow>$$

$$|1,-1> = |\downarrow\downarrow>$$

Summary for Spin ½ + Spin ½

$j = 0$ (singlet):

$$|0,0> = \frac{1}{\sqrt{2}}|\uparrow,\downarrow> - \frac{1}{\sqrt{2}}|\downarrow,\uparrow>$$

Antisymmetric

$j = 1$ (triplet):

$$|1,1> = |\uparrow\uparrow>$$

$$|1,0> = \frac{1}{\sqrt{2}}|\uparrow\downarrow> - \frac{1}{\sqrt{2}}|\downarrow\uparrow>$$

Symmetric

$$|1,-1> = |\downarrow\downarrow>$$

Pauli exclusion principle

The *total* wavefunction of a system of *Fermions* must be *antisymmetric*.

For two electrons, under LS coupling:

$$\Psi_{\text{Total}} = \psi_{\text{spatial}}\, \chi_{\text{spin}}$$

Ψ_{Total} has to be antisymmetric (Pauli exclusion principle):

If ψ_{spatial} is symmetric (ℓ = even) then χ_{spin} is antisymmetric (singlet)

If ψ_{spatial} is antisymmetric (ℓ = odd) then χ_{spin} is symmetric (triplet)