Class 23: Work and kinetic energy (Con’t)

Acceleration by chain rule (1D)

If we know the velocity as a function of time, we can differentiate it w.r.t. time and find out how the acceleration depends on time:

$$a_x = \frac{dv}{dt}$$

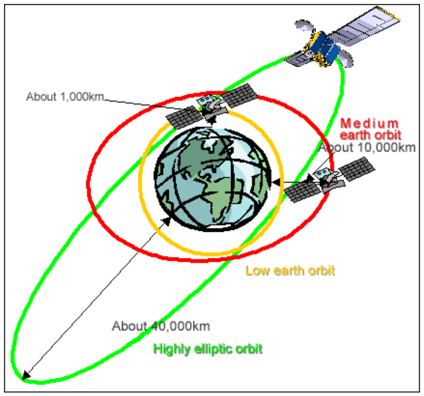
However, very often we only know the velocity as a function of position (i.e. coordinate x). What to do in this case?

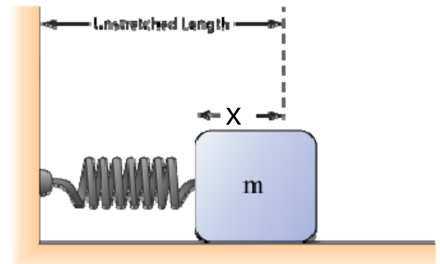
$$a_x = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx}$$

$$\therefore \boxed{a_x = v\frac{dv}{dx}}$$

The answer

In most cases we live in a "force field" – there is always a force acting on us and this force depends on where we are.





$$\Sigma F_x = m \frac{d^2}{dt^2} x \quad \Rightarrow \Sigma F_x = m \frac{d}{dx} v_x$$

$$\Rightarrow m\, v_x \frac{d}{dx} v_x = F_x$$

$$\Rightarrow \left(\frac{1}{2} m v_{xf}^2 - \frac{1}{2} m v_{xi}^2 \right) = \int_{x_i}^{x_f} F_x \, dx$$

3D

$$\left(\frac{1}{2}mv_{fx}^2 - \frac{1}{2}mv_{ix}^2\right) = \int_{x_i}^{x_f} F_x\, dx$$

$$\left(\frac{1}{2}mv_{fy}^2 - \frac{1}{2}mv_{iy}^2\right) = \int_{y_i}^{y_f} F_y\, dy$$

$$\left(\frac{1}{2}mv_{fz}^2 - \frac{1}{2}mv_{iz}^2\right) = \int_{z_i}^{z_f} F_z\, dz$$

+

$$\left(\frac{1}{2}mv_{f}^2 - \frac{1}{2}mv_{i}^2\right) = \left(\int_{x_i}^{x_f} F_x\, dx + \int_{y_i}^{y_f} F_y\, dy + \int_{z_i}^{z_f} F_z\, dz\right)$$

Work (abbreviation: W)

Work done W by a force $\vec{F} = \int_{x_i}^{x_f} F_x \, dx + \int_{y_i}^{y_f} F_y \, dy + \int_{z_i}^{z_f} F_z \, dz$

1. Work is a scalar (sum of definite integrals) – it has no direction.

2. Unit of work: Joule (J). Joule is not a fundamental unit, $J \equiv Nm \equiv Kgm^2s^{-2}$.

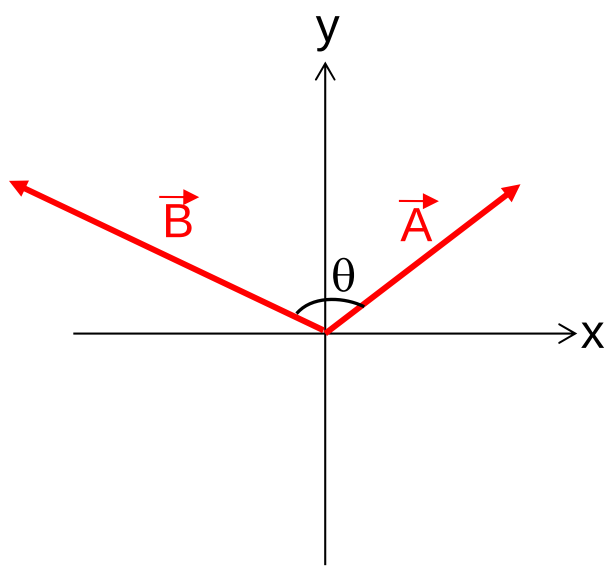
3. Work done by a force can be positive, negative, or 0.

Dot product (a.k.a. scalar product)

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\Updownarrow$$

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$



1. The result is a scalar, that's why its called the scalar product.

2. The equivalency is useful to calculate the angle between two vectors, if you know the components of these two vectors.

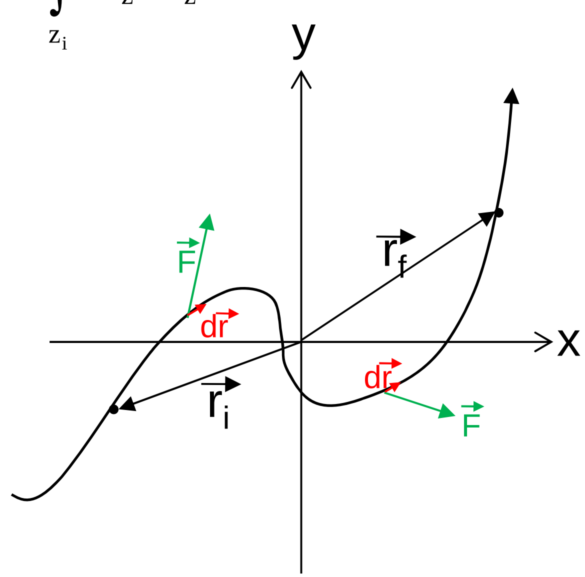
Work

Work done W by a force $\vec{F} = \int_{x_i}^{x_f} F_x \, dx + \int_{y_i}^{y_f} F_y \, dy + \int_{z_i}^{z_f} F_z \, dz$

$$\vec{F} = \int_{x_i}^{x_f} F_x \, dx + \int_{y_i}^{y_f} F_y \, dy + \int_{z_i}^{z_f} F_z \, dz$$

$$= \int_{x_i}^{x_f} F_x \, dr_x + \int_{y_i}^{y_f} F_y \, dr_y + \int_{z_i}^{z_f} F_z \, dr_z$$

Work done W by a force $\vec{F} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$



When $\vec{F}$ is constant

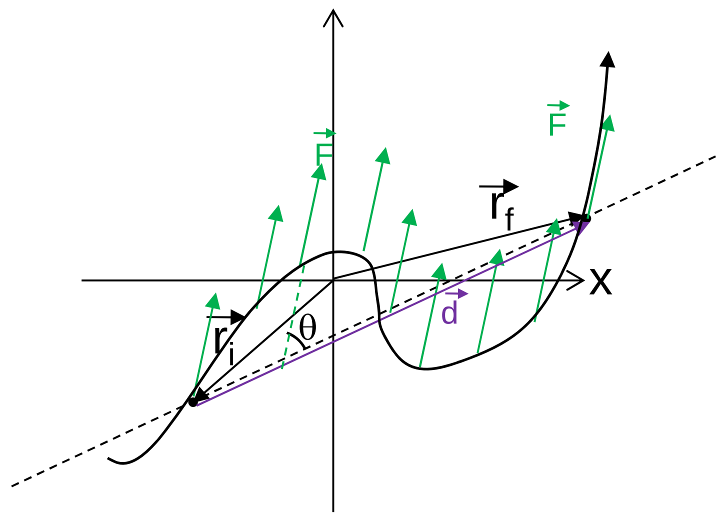
Work done W by force $\vec{F} = \int_{x_i}^{x_f} F_x \, dx + \int_{y_i}^{y_f} F_y \, dy + \int_{z_i}^{z_f} F_z \, dz$

$$= F_x \int_{x_i}^{x_f} dx + F_y \int_{y_i}^{y_f} dy + F_z \int_{z_i}^{z_f} dz$$

$$= F_x \Delta x + F_y \Delta y + F_z \Delta z$$

$$= \boxed{\vec{F} \cdot \vec{d}} \qquad (\vec{d} = \Delta x \hat{i} + \Delta y \hat{j} + \Delta z \hat{k} = \vec{r}_f - \vec{r}_i)$$

Path independent



More than one force

$$\left(\frac{1}{2}mv_{fx}^2 - \frac{1}{2}mv_{ix}^2\right) = \int_{x_i}^{x_f} F_x\, dx$$

$$\left(\frac{1}{2}mv_{fy}^2 - \frac{1}{2}mv_{iy}^2\right) = \int_{y_i}^{y_f} F_y\, dy$$

$$\left(\frac{1}{2}mv_{fz}^2 - \frac{1}{2}mv_{iz}^2\right) = \int_{z_i}^{z_f} F_z\, dz$$

+

$$\left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\right) = \left(\int_{x_i}^{x_f} F_x\, dx + \int_{y_i}^{y_f} F_y\, dy + \int_{z_i}^{z_f} F_z\, dz\right)$$

↓ Dot product notations

One force: $$\left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\right) = \int_{x_i}^{x_f} \vec{F}\cdot d\vec{r}$$

Many forces: $$\left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\right) = \sum_i \int_{x_i}^{x_f} \vec{F}_i \cdot d\vec{r} = \int_{x_i}^{x_f} \left(\sum_i \vec{F}_i\right)\cdot d\vec{r}$$

↑ Total work ↑ Work done by total force

Kinetic energy (abbreviation: K)

$$\text{Kinetic energy of a moving particle} = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}mv^2 = \frac{1}{2}m\,\vec{v}\cdot\vec{v}$$

1. Kinetic energy is a scalar – it has no direction.

2. Unit of kinetic energy: Joule (J), the same unit as work.

3. Kinetic energy is always positive, because $m>0$ and $v^2>0$. There is no negative kinetic energy.