Report on a simplified derivation of the overlap Dirac operator due to Fosco and Torroba

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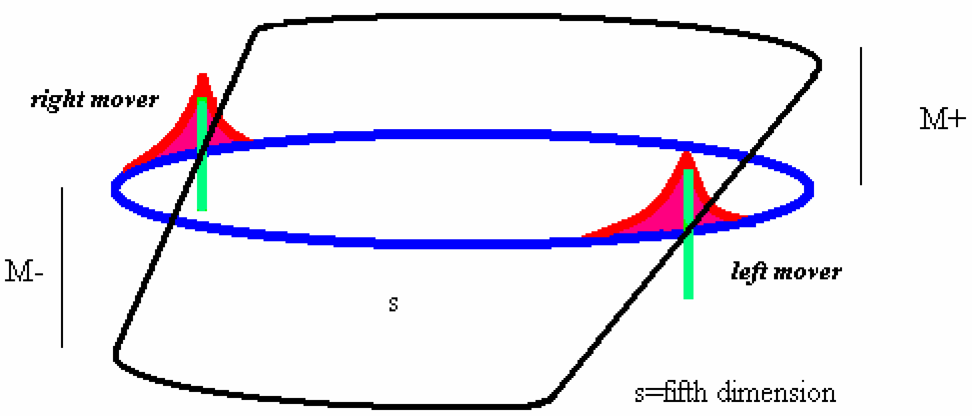
Review I

The Callan and Harvey construction led to David Kaplan's original paper. There we had:

1) There is a $A_5(x_\mu , s)$.
2) $A_\mu(x_\nu,s)$ depends on s

FIG. 1



Review II

There also was a paper by Slavnov+Frolov, using 4D gauge theory, but with what amounts to an infinite flavor space.

NN: Kaplan and Slavnov+Frolov are the same if:

0) We send one wall to infinity, keeping only the vicinity of the other wall finite.

1) and we set:

$$A_5 = 0, \frac{\partial A_\mu}{\partial s} = 0$$

Review III

2) The 4D action, with s viewed as a continuous or discrete flavor space has the structure:

$$\gamma_\mu D_\mu + M$$

where M has an index

$$MM^\dagger > 0, ...dim(Ker(M)) = 1$$

Review IV

A) This explains stability against radiative corrections on the lattice.

B) This prevents the extra gauge degrees of freedom from creating extra walls.

Kaplan, independently also suggested 1)

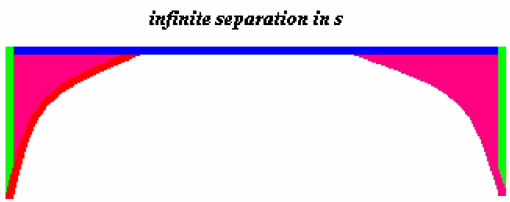
Review V

Using this scheme one proceeds to the vector-like case:

0) Make wall separation infinite, but now keep both wall vicinities.

1) Take M_+ to infinity --can ignore half of the circle.

Fig. 2



Review VI

This produces the domain wall fermions.

Shamir arrives at 1) directly, without introducing first a finite M_+ followed by taking M_+ to infinity. That these two ways are equivalent follows from replaying the derivation of MIT bag boundary conditions in the 70's.

Review VIII

There are 2 options now:

1) Work out analytically the infinite wall separation case (NN,N) and implement the result numerically → overlap operator:

$$D_o = \frac{1+\gamma_5 \mathrm{sign}(H_W)}{2}$$

Review IX

2) Latticize the 5-th dimension and stick to the 5D Dirac operator which is simple.

In case 1) we aim for chiral symmetry to machine accuracy. In case 2) chiral symmetry is broken and we need to characterize the amount of breaking.

This talk is concerned with option 1).

Starting and end points

- The Weyl components of the Dirac fermion live on separate 4D worlds; they get identified after the overlap formula is obtained.
- The overlap formula has a funny square root hidden under the usage of the sign function.

We wish to find a simple derivation that produces the square roots and keeps the two parts of the massless Dirac fermion at identical space time points. We can achieve this, but the derivation is somewhat non-rigorous.

A delta function singularity

$$\mathcal{D}_{d+1} = \gamma_s \partial_s + \not{D} + m + \xi \delta(s)$$

- ξ is dimensionless.
- Only the four dimensional gauge background appears.
- The delta function is the limit of two infinite jumps, one up followed by one down.

$$\mathcal{D}_{d+1} = \mathcal{D} + \widehat{M}, \quad < x, s | \widehat{M} | x' s' > = \xi \delta^4 (x - x') \delta(s) \delta(s')$$

The propagator at s=0

$$\mathcal{K} = \int_{-\infty}^{+\infty} \frac{dk_s}{2\pi} \frac{1}{i\gamma_s k_s + \not{D} + m}$$

Here we use a convention with $A_\mu = -A_\mu^\dagger$ making $\not{D}$ antihermitian and m real. $\mathcal{K}$ is an operator on fermions on $\mathcal{M}$, the four dimensional space at $s = 0$.

The derivation

Do a loop expansion in ξ – at n-th order you have

$$\Gamma^{(n)}_{\mathcal{O}}(A) = \frac{(-1)^n}{n}\,\mathrm{Tr}_{d+1}\left[(\mathcal{D}^{-1}\,\widehat{M})^n\right] \qquad \Gamma^{(n)}_{\mathcal{O}}(A) = \frac{(-1)^n}{n}\,\xi^n\,\mathrm{Tr}_d\left[\mathcal{K}^n\right]$$

$$\mathcal{K} = \int_{-\infty}^{+\infty}\frac{dk_s}{2\pi}\left(\frac{-i\gamma_s k_s - \not{D} + m}{k_s^2 - \not{D}^2 + m^2}\right) = (-\not{D}+m)\int_{-\infty}^{+\infty}\frac{dk_s}{2\pi}\,\frac{1}{k_s^2 - \not{D}^2 + m^2}$$

$$\mathcal{K} = \frac{1}{2}V = \frac{1}{2}\frac{-\not{D}+m}{\sqrt{-\not{D}^2+m^2}} \qquad \Gamma_{\mathcal{O}}(A) = -\,\mathrm{Tr}_d\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,(\xi\mathcal{K})^n\right]$$

$$\Gamma_{\mathcal{O}}(A) = -\,\mathrm{Tr}_d\ln\left[1+\frac{\xi}{2}\,V\right] = -\,\ln\det_d\left[1+\frac{\xi}{2}\,V\right]$$

$$e^{-\Gamma_{\mathcal{O}}(A)} \equiv \det_d\mathcal{O} \qquad \mathcal{O} = 1+\frac{\xi}{2}\,V$$

The value of ξ

- Require that $\mathcal{D}_{d+1}$ have zero modes in an instanton background.
- Define singularity by

$$\Psi(x,0) = \tfrac{1}{2}[\Psi(x,0+) + \Psi(x,0-)]$$

- Obtain from

$$[\pm\partial_s + m + \widehat{M}]\Psi = 0 \text{ that } \xi = -2\,\mathrm{sign}(m)$$

- Set m>0 and ξ=-2.

Field content I

Decouple the Ψ fields at s=0 by an auxiliary four dimensional field, χ :

$$e^{-\int d^dx ds \bar{\Psi}\widehat{M}\Psi} =$$

$$\frac{\int \mathcal{D}\chi\,\mathcal{D}\bar{\chi}\; e^{-\int d^dx\left(\frac{1}{\xi}\bar{\chi}(x)\chi(x) - i[\bar{\chi}(x)\Psi(x,0) + \bar{\Psi}(x,0)\chi(x)]\right)}}{\int \mathcal{D}\chi\,\mathcal{D}\bar{\chi}\; e^{-\int d^dx\frac{1}{\xi}\bar{\chi}(x)\chi(x)}}$$

Field content II

- Calculate the two by two propagator matrix for $\Psi(x,0)$ and χ (x) and their barred counterparts.
- Form linear combinations $\Phi_\alpha = \Psi(x,0) - \imath\alpha\chi(x)$ and identical for the barred partners.
- For $\alpha = -\frac{1\pm\sqrt{2}}{2}$ Φ_α has a propagator that anti-commutes with γ_5 .
- For $\alpha = \frac{1}{2}$ Φ_α has a propagator that is proportional to δ^4(x-y), - that is this linear combination does not propagate.
- The above two types of fields mix.

Summary

- A delta function singularity in the fifth dimension, with a specific strength produces massless fermions propagating along the four dimensional wall.
- The field content is that of an exactly massless Dirac fermion that mixes with a non-propagating field.
- The effective action associated with the defect is the determinant of the overlap Dirac operator.
- Although somewhat non-rigorous, this setup produces a simple derivation of the overlap Dirac operator.

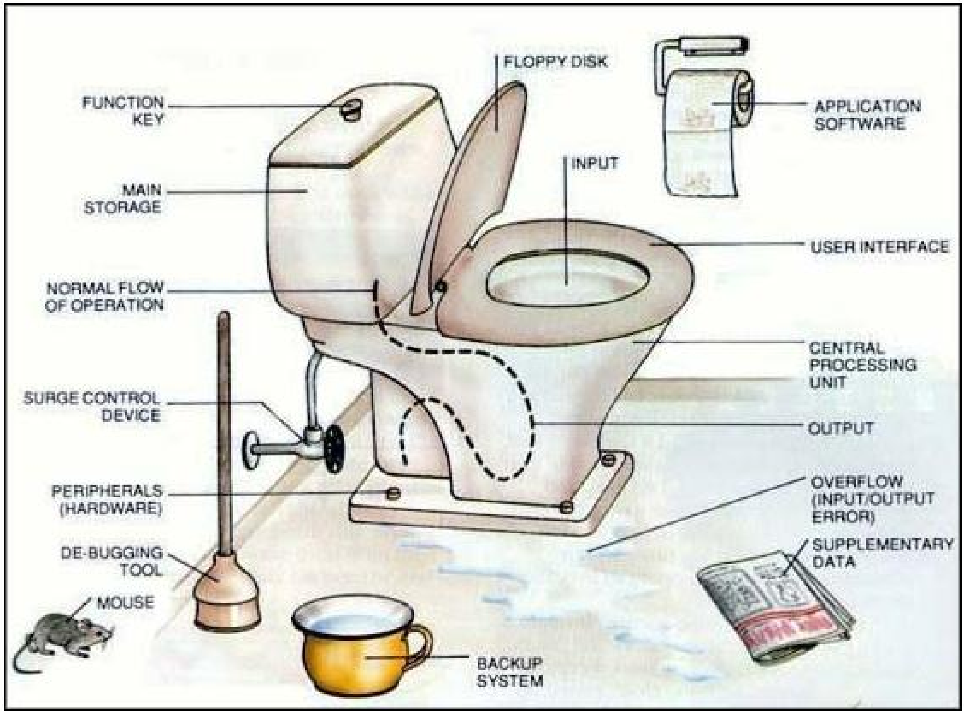
And what about computer time ?

As you know, the overlap charges a high cost for generating a statistically independent gauge configuration in a dynamical simulation and even for propagator calculation. Keh-Fei has shown that the cost per unit of physics is not really high in comparison – this fact has not yet permeated the consciousness of the people in our field who control computer resources and the wrong beans are still being the ones that get counted.

To prove that we are not oblivious to computational cost I pilfered a one slide summary of the essence of computation.

What is a computer and what does one do with it ?

Do you understand computer technology? Check your knowledges!



CONCLUSION