Collisions in 2D

As usual, we ignore friction during the collision because the impulse due to collisional forces exceeds by several orders of magnitude the impulse due to friction.

We have two extreme kinds of collisions.

The totally inelastic collision where the two objects stick together after the collision and move as one unit, and the perfectly elastic collision, in which the total KE is also conserved.

Recall that $\boxed{\text{if} \quad \vec{F}_{ext} = 0 \quad \vec{P}_{tot} \text{ is conserved.}}$ (1)

For a totally inelastic collision

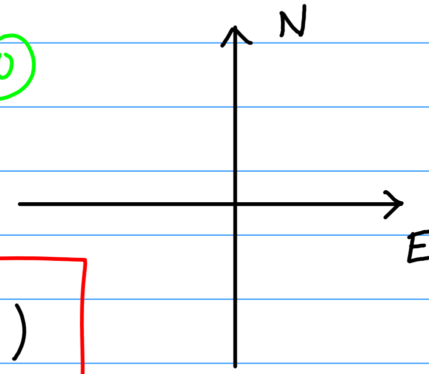
$$\boxed{\vec{P}_{i,tot} = M_1\vec{v}_{1i} + M_2\vec{v}_{2i}} \quad (2)$$

$$\boxed{\vec{P}_{f,tot} = (M_1+M_2)\,\vec{v}_f} \quad (3)$$

So $$\boxed{\vec{v}_f = \frac{M_1\vec{v}_{1i} + M_2\vec{v}_{2i}}{M_1+M_2}} \quad (4)$$

(5)

Example: A car of mass $\boxed{M_1 = 1500\,kg}$ going due East at $\boxed{20\,m/s}$ (6) hits a truck of mass $\boxed{M_2 = 10000\,kg}$ (7) going South at $\boxed{15\,m/s.}$ (8) If the collision takes $\boxed{0.1\,sec}$ (9) find the average accelerations suffered by the two drivers.



$$\boxed{\vec{p}_{1i} = M_1 \vec{v}_{1i} = 1500\,\text{kg} \times 20\,\frac{\text{m}}{\text{s}}\,\hat{i} = 30000\,\text{kg}\,\frac{\text{m}}{\text{s}}\,\hat{i}}$$ (10)

$$\boxed{\vec{p}_{2i} = M_2 \vec{v}_{2i} = 10000\,\text{kg}\left(-15\,\frac{\text{m}}{\text{s}}\,\hat{j}\right) = -150000\,\text{kg m/s}\,\hat{j}}$$ (11)

$$\boxed{\vec{P}_{i,tot} = 10^4\,\text{kg m/s}\,[3\hat{i} - 15\hat{j}]}$$ (12)

(13)

$$\boxed{\vec{P}_{f,tot} = (1500\,\text{kg} + 10000\,\text{kg})\,\vec{v}_f = 10^4 \times 1.15\,\text{kg}\,\vec{v}_f}$$

Conservation of momentum implies

$$10^4\,\text{kg}\,\frac{\text{m}}{\text{s}}\,[3\hat{i} - 15\hat{j}] = 10^4\,\text{kg} \times 1.15\,\vec{v}_f$$

(14)

$$\boxed{\vec{v}_f = \frac{1}{1.15}\,[3\hat{i} - 15\hat{j}]\,\text{m/s} = 2.61\,\frac{\text{m}}{\text{s}}\,\hat{i} - 13.04\,\frac{\text{m}}{\text{s}}\,\hat{j}}$$

To find the average accelerations we use

$$\boxed{\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t}}$$ (15)

For the car $\quad \vec{a}_{av,c} = \dfrac{\vec{v}_f - \vec{v}_{i1}}{\Delta t} = \dfrac{2.61\hat{i} - 13.04\hat{j} - 20\hat{i}}{0.1}$

$$\boxed{\vec{a}_{av,c} = (-173.9\,\hat{i} - 130.4\,\hat{j})\,\text{m/s}^2}$$ (16)

$$\boxed{|\vec{a}_{av,c}| = \sqrt{(173.9)^2 + (130.4)^2} = 217.4\ m/s^2} \quad (17)$$

about 22 g's!!

For the truck

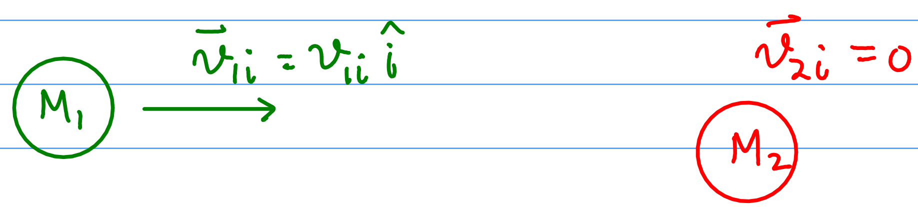
$$\vec{a}_{av,T} = \frac{\vec{v}_f - \vec{v}_{i2}}{\Delta t} = \frac{2.61\ \frac{m}{s}\ \hat{i} - 13.04\ \frac{m}{s}\ \hat{j} + 15\ \frac{m}{s}\ \hat{j}}{0.1}$$

$$\boxed{\vec{a}_{av,T} = 26.1\ \frac{m}{s^2}\ \hat{i} + 19.6\ \frac{m}{s^2}\ \hat{j}} \quad (18)$$

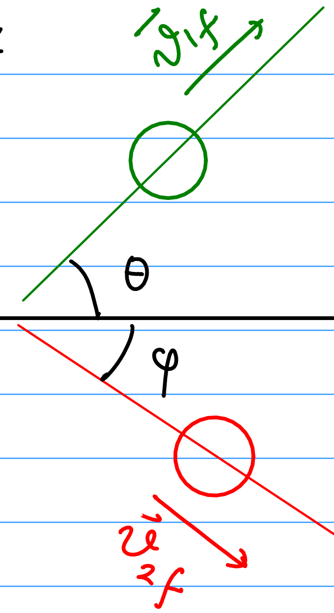
$$\boxed{|\vec{a}_{av,T}| = 32.6\ m/s^2} \quad (19)$$

about $3\frac{1}{2}$ g's.

Now consider perfectly elastic collisions. Take the simple case when one object, say M_2, is initially at rest, and choose axes such that the initial velocity of M_1 is along $\hat{i}$



After the collision, we have



Conservation of momentum implies

$$\boxed{M_1 v_{1i}\,\hat{i} = M_1 \vec{v}_{1f} + M_2 \vec{v}_{2f}} \quad (20)$$

or

$$\boxed{\begin{aligned} &= M_1 v_{1f} \cos\theta\,\hat{i} + M_1 v_{1f} \sin\theta\,\hat{j} \\ &\quad + M_2 v_{2f} \cos\varphi\,\hat{i} - M_2 v_{2f} \sin\varphi\,\hat{j} \end{aligned}} \quad (21)$$

Since momentum conservation is a vector eq^n each component is separately conserved.

$$\boxed{\begin{aligned} M_1 v_{1i} &= M_1 v_{1f} \cos\theta + M_2 v_{2f} \cos\varphi \\ 0 &= M_1 v_{1f} \sin\theta - M_2 v_{2f} \sin\varphi \end{aligned}} \quad (22)$$

Conservation of KE implies

$$\boxed{\frac{1}{2} M_1 v_{1i}^2 = \frac{1}{2} M_1 v_{1f}^2 + \frac{1}{2} M_2 v_{2f}^2} \quad (23)$$

We see that there are 4 unknowns, v_{1f}, v_{2f}, θ and φ, and only 3 eq^{n}s. So we need to know one of the 4 before we can proceed.

There are several possibilities. We could be given one of the angles θ, φ or we could be given one of the final speeds. Let's see how these work out.

We start with $\boxed{M_1 = 2\,kg}$ (24) $\boxed{M_2 = 3\,kg}$ (25)

$$\boxed{\vec{v}_{1i} = 10\, m/s\ \hat{i}} \quad (26)$$

Example 1: Given $\boxed{\theta = 30^\circ}$ (27)

$$\boxed{p_{i,tot,x} = M_1 v_{1i} = 20\, kg\, m/s} \quad (28)$$

$$p_{f,tot,x} = M_1 v_{1f} \cos\theta + M_2 v_{2f} \cos\varphi$$

$$\Rightarrow \boxed{M_2 v_{2f} \cos\varphi = M_1 v_{1i} - M_1 v_{1f} \cos\theta} \quad (29)$$

$$0 = p_{f,tot,y} = M_1 v_{1f} \sin\theta - M_2 v_{2f} \sin\varphi$$

$$\Rightarrow \boxed{M_2 v_{2f} \sin\varphi = M_1 v_{1f} \sin\theta} \quad (30)$$

Square (29), (30) and add using $\sin^2\varphi + \cos^2\varphi = 1$

$$\boxed{\begin{aligned} M_2^2 v_{2f}^2 &= M_1^2 v_{1f}^2 \sin^2\theta + (M_1 v_{1i} - M_1 v_{1f} \cos\theta)^2 \\ &= M_1^2 (v_{1i}^2 + v_{1f}^2 - 2 v_{1i} v_{1f} \cos\theta) \end{aligned}} \quad (31)$$

or $$\boxed{\frac{1}{2} M_2 v_{2f}^2 = \frac{M_1^2}{2 M_2} (v_{1i}^2 + v_{1f}^2 - 2 v_{1i} v_{1f} \cos\theta)} \quad (32)$$

Use KE conservation

$$\boxed{\frac{1}{2} M_1 v_{1i}^2 = \frac{1}{2} M_1 v_{1f}^2 + \frac{1}{2} M_2 v_{2f}^2} \quad (33)$$

(34)

or $$\boxed{M_1 v_{1i}^2 = M_1 v_{1f}^2 + \frac{M_1^2}{M_2}\left(v_{1i}^2 + v_{1f}^2 - 2 v_{1i} v_{1f} \cos\theta\right)}$$

The only unknown is v_{1f}. In our example, putting in the numbers

$$2 \times (10)^2 = 2 \times v_{1f}^2 + \frac{4}{3}\left(10^2 + v_{1f}^2 - 2(10)\, v_{1f} \frac{\sqrt{3}}{2}\right)$$

or $$\frac{10}{3} v_{1f}^2 - \frac{40}{\sqrt{3}} v_{1f} - \frac{200}{3} = 0$$

$$\boxed{v_{1f}^2 - 4\sqrt{3}\, v_{1f} - 20 = 0} \quad (35)$$

$$v_{1f} = \frac{4\sqrt{3} \pm \sqrt{48+80}}{2} = \frac{4\sqrt{3} \pm 8\sqrt{2}}{2}$$

v_{1f} has to be >0

$$\boxed{v_{1f} = 2\sqrt{3} + 4\sqrt{2} = 9.121 \, m/s} \quad (36)$$

Now go back to (33)

$$2\,(10)^2 = 2(9.121)^2 + 3(v_{2f})^2$$

or $\boxed{v_{2f} = 3.35\ m/s}$ 37

Now go to 30

$$\boxed{3\,(3.35)\,\sin\varphi = 2(9.121)\,\frac{1}{2}} \quad 38$$

$$\boxed{\sin\varphi = 0.9076} \quad 39 \Rightarrow \boxed{\begin{aligned}\varphi &= 1.137\ \text{rad} \\ &= 65.2^\circ\end{aligned}} \quad 40$$

Example 2 : Instead of θ, suppose it is known that $\boxed{v_{1f} = 7\,m/s.}$ 41 Then, 1st go to KE conservation

$$\boxed{2\,(10)^2 = 2(7)^2 + 3(v_{2f})^2} \quad 42$$

$$\Rightarrow \quad v_{2f}^2 = \frac{2}{3}(51) = 34\left(\frac{m}{s}\right)^2$$

$$\Rightarrow \quad \boxed{v_f = 5.831\ m/s.} \quad 43$$

Now we 31

$$\boxed{M_2^2 v_{2f}^2 = M_1^2\left(v_{1i}^2 + v_{1f}^2 - 2v_{1i}\, v_{1f} \cos\theta\right)} \quad 44$$

$$\text{or} \quad \frac{(3)^2(5.831)^2}{(2)^2} = 10^2 + 7^2 - 140\cos\theta$$

$$\Rightarrow \quad \boxed{\cos\theta = 0.518} \quad 45 \quad \Rightarrow \boxed{\begin{aligned}\theta &= 1.026\ \text{rad} \\ &= 58.8^\circ\end{aligned}} \quad 46$$

Finally, we (30) to find φ

$$(3)(5.831)\sin\varphi = (2)(7)\sin(58.8^\circ)$$

$$\boxed{\sin\varphi = 0.6845} \quad (47) \Rightarrow \boxed{\begin{aligned}\varphi &= 0.754 \text{ rad} \\ &= 43.2^\circ\end{aligned}} \quad (48)$$

Here is an interesting fact. If $M_1 = M_2$, then $\theta + \varphi = \frac{\pi}{2}$. Let us show this.

$$\boxed{\begin{aligned} M v_{1i} &= M v_{1f}\cos\theta + M v_{2f}\cos\varphi \\ 0 &= M v_{1f}\sin\theta - M v_{2f}\sin\varphi \end{aligned}} \quad (49)$$

Square and add

$$M^2 v_{1i}^2 = M^2\left\{(v_{1f}\cos\theta + v_{2f}\cos\varphi)^2 + (v_{1f}\sin\theta - v_{2f}\sin\varphi)^2\right\} \quad (50)$$

$$\boxed{\begin{aligned} v_{1i}^2 &= v_{1f}^2 + v_{2f}^2 + 2v_{1f}v_{2f}[\cos\theta\cos\varphi - \sin\theta\sin\varphi] \\ &= v_{1f}^2 + v_{2f}^2 + 2v_{1f}v_{2f}\cos(\theta+\varphi) \end{aligned}}$$

But by KE conservation

$$\boxed{v_{1i}^2 = v_{1f}^2 + v_{2f}^2} \quad (51) \Rightarrow \boxed{\cos(\theta+\varphi) = 0 \quad \theta + \varphi = \frac{\pi}{2}} \quad (52)$$