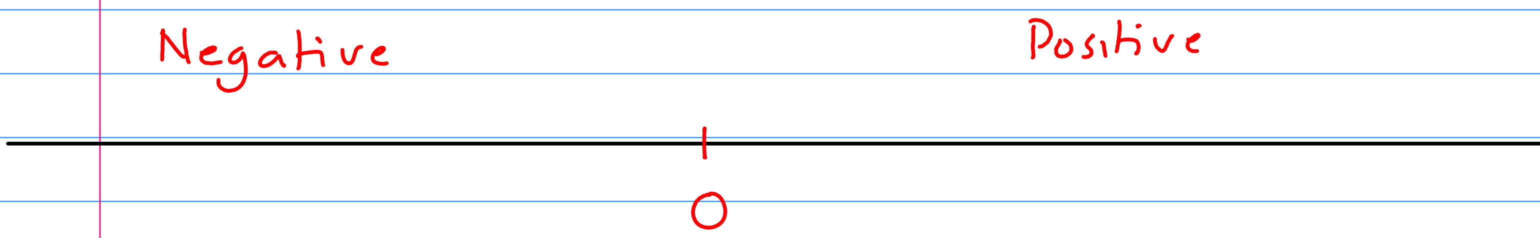
One-dimensional motion

The goal of this class is to be able to describe motion in 1 dimension, that is, motion along a straight line.

The very 1st thing we have to do is choose an origin somewhere on this straight line, and the direction we call positive

These choices, origin and direction, are entirely up to us. Another person might choose the origin in a different place, and one of our tasks will be to translate between our measurements and hers.



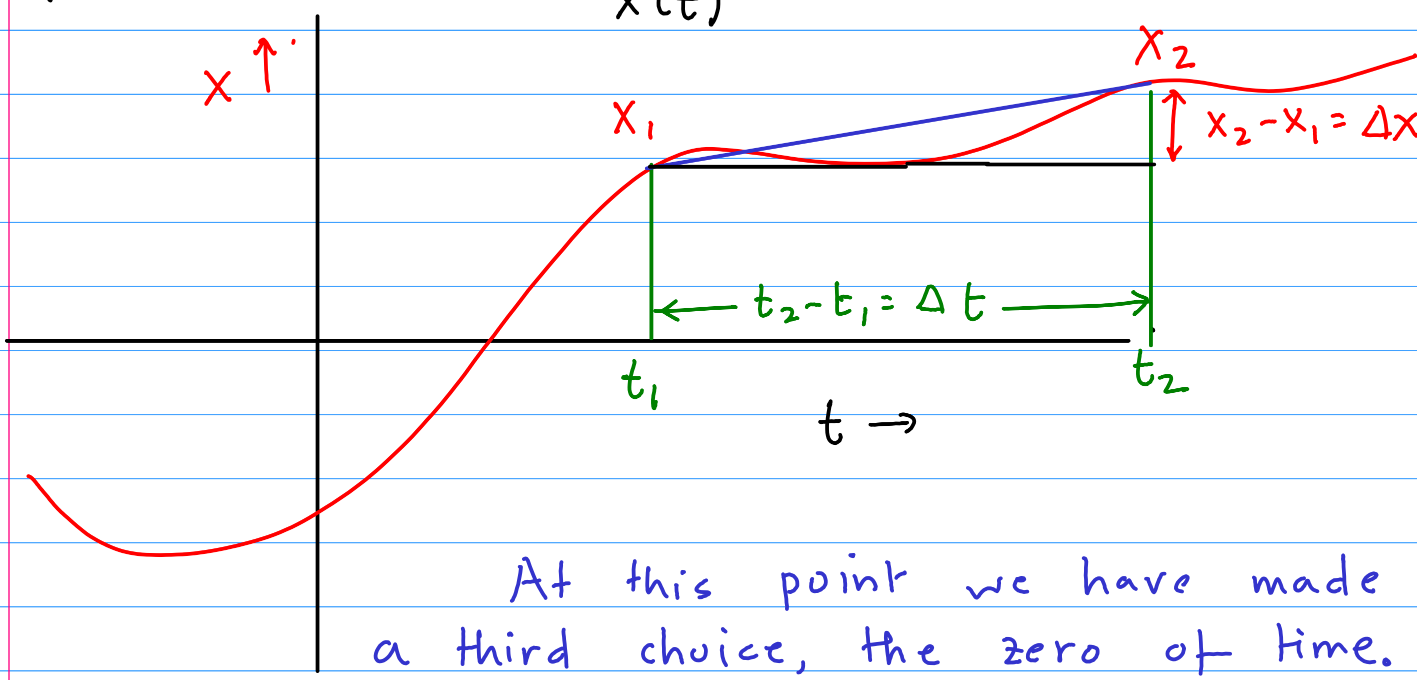
Another choice I will disuss towards the end has to do with whether the two people making measurements, Alice (A) and Bob (B) are stationary with respect to each other. It could happen that Bob is moving at constant velocity w.r.t. Alice. Thinking deeply about this led Galileo and Einstein to their versions of Relativity.

So, the 1st quantity we must measure to describe motion is the position of a body.

A physical body has nonzero size, so it doesn't have a single position. For our present purposes, let us mark a spot on the body (whose size determines the presicion of our position measurement) and idealize the position of the body as the position of the spot.

The position will then be some measured function of time

$x(t)$



At this point we have made a third choice, the zero of time. This could also be different between Alice and Bob.

The positions at two arbitrary times t_1 and t_2 are $x_1 = x(t_1)$ and $x_2 = x(t_2)$

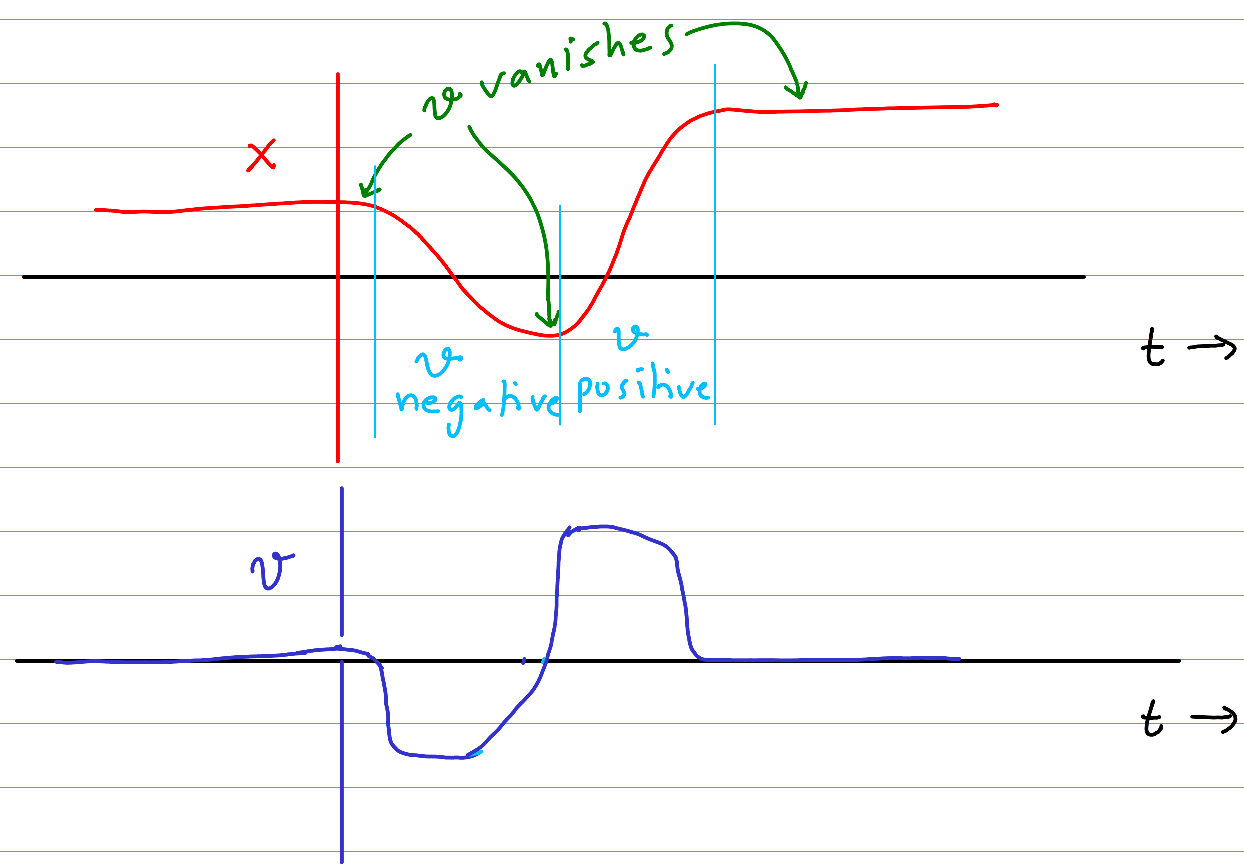
The average velocity between t_1 & t_2 is

$$\boxed{v_{av}(t_1;t_2) = \frac{x_2-x_1}{t_2-t_1} = \frac{\Delta x}{\Delta t}} \quad (1)$$

You are familiar from calculus with the notion of a limit. The instantaneous velocity at time t is defined as

$$\boxed{v(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}} \quad (2)$$

The instantaneous velocity is the rate of change of position, which is also the slope of the x vs t graph



In order to understand motion we need to go one step further: From velocity to acceleration.

Once again if the instantaneous velocity at t_1 is $v(t_1) = v_1$ and that at t_2 is $v(t_2) = v_2$ the average acceleration between t_1, t_2 is

$$\boxed{a_{av}(t_1; t_2) = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}} \quad (3)$$

The instantaneous acceleration is (4)

$$\boxed{a(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2}}$$

Of course, $a(t)$ is the slope of the v vs t graph at t.

Can one go "backwards", from $a \to v$ or from $v \to x$? Up to an "initial condition" we can.

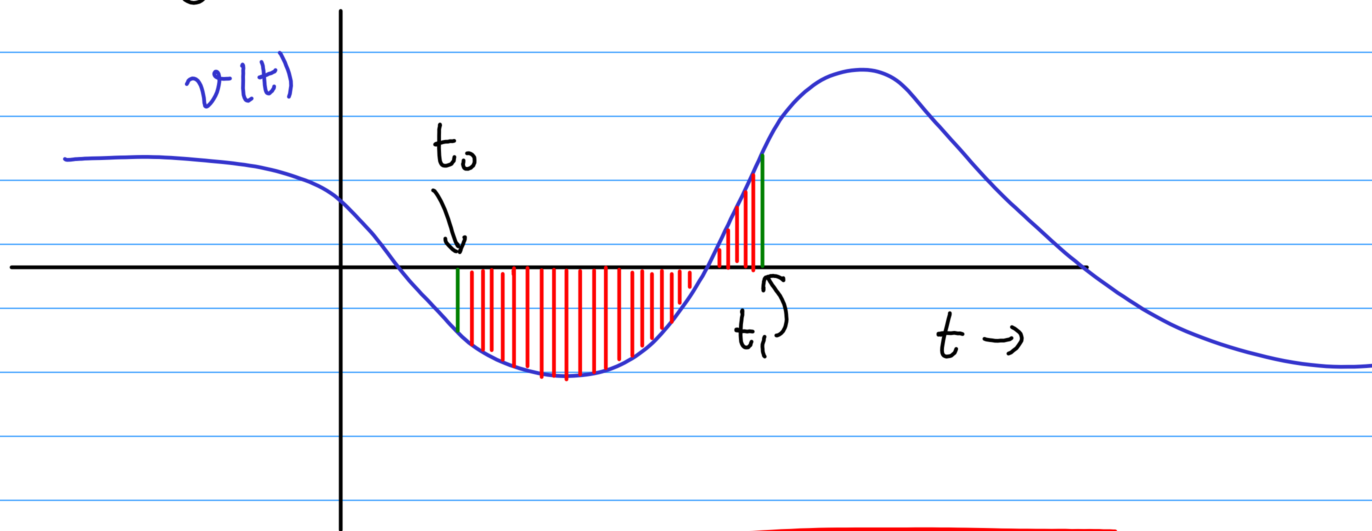
The fundamental theorem of Calculus says that differentiation and integration are inverse operations

$$\boxed{v(t) = \frac{dx(t)}{dt} \Rightarrow \quad x(t) = \int^{t} dt' \, v(t')} \quad (5)$$

More presicely, if we know $x(t_0) = x_0$ for some time t_0, then

$$\boxed{x(t_1) = x(t_0) + \int_{t_0}^{t_1} dt\, v(t)} \qquad (6)$$

This should be visualized as the area under the curve of $v(t)$, with the convention that in a region where $v < 0$ the area is negative



Similarly

$$\boxed{v(t_1) = v(t_0) + \int_{t_0}^{t_1} dt\, a(t)} \qquad (7)$$

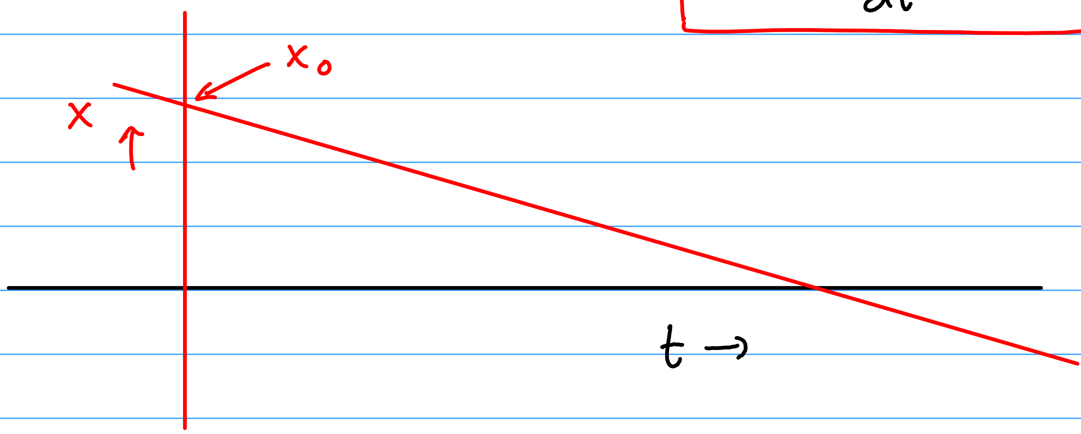
So far we have been very general. Let us now consider two important special cases
(i) Motion with constant velocity and
(ii) Motion with constant acceleration

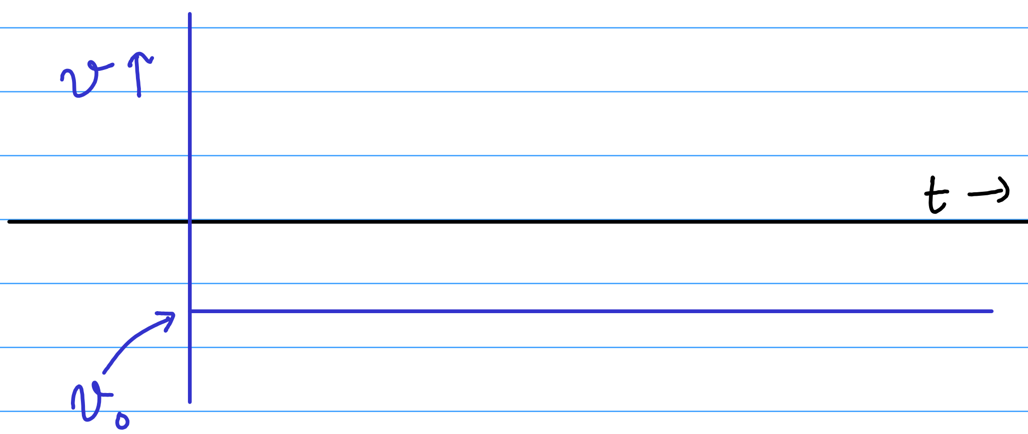
One-dimensional motion with constant v

We know that $\boxed{v(t) \equiv v(0) = v_0 \text{ (say)}}$ (8)

$$\Rightarrow \boxed{x(t) = x(0) + \int_0^t v_0\, dt' = x(0) + vt} \quad (9)$$

and of course $\boxed{a(t) = \frac{dv_0}{dt} = 0}$ (10)





In the above example $x_0 > 0$ but $v_0 < 0$.

We will learn that such a motion occurs when the net force on a body vanishes.

One-dimensional motion with constant acceleration

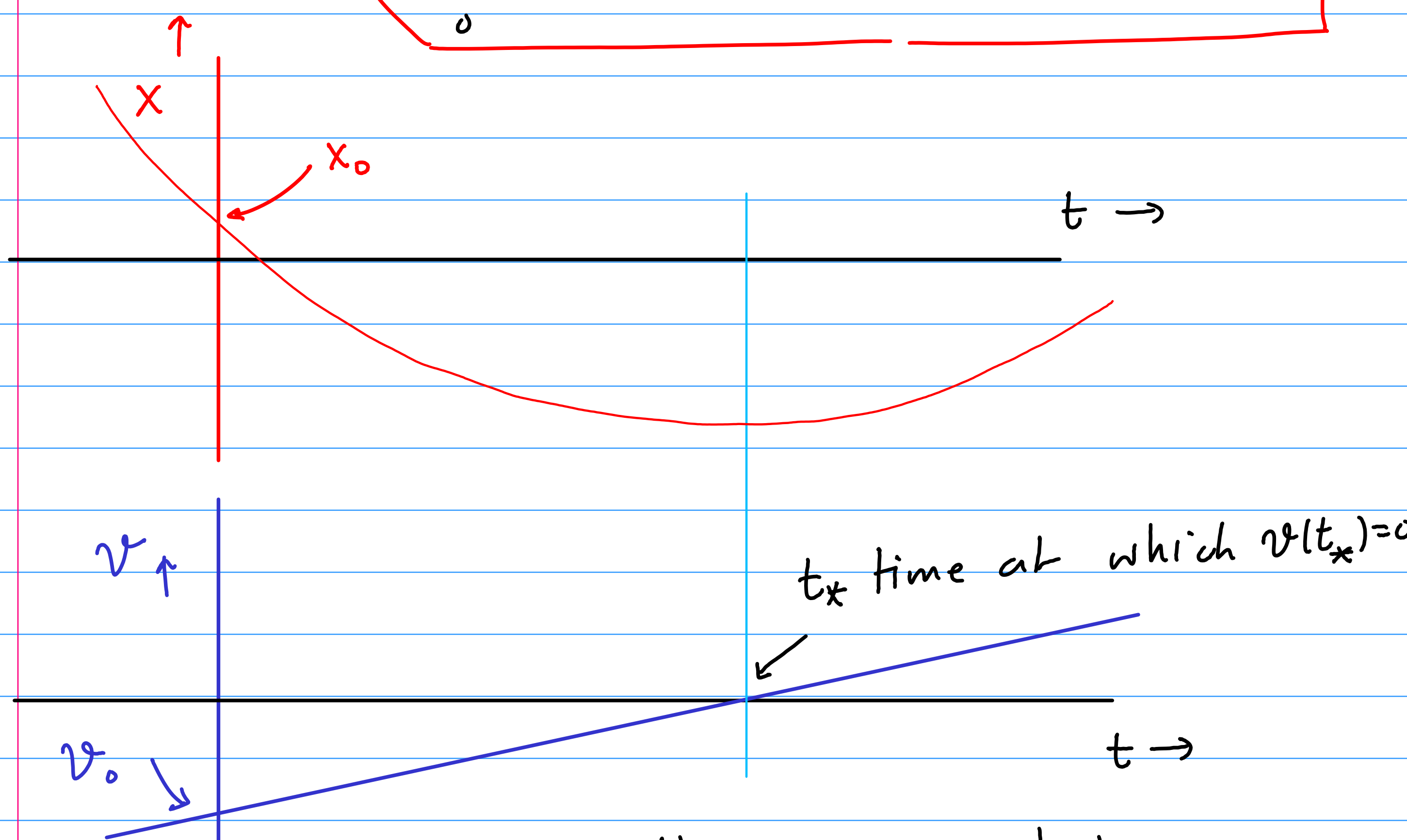
Now $$\boxed{a(t) \equiv a(0) = a_0 \quad (\text{say})} \tag{11}$$

We proceed "backwards" by integrating

$$\boxed{v(t) = v(0) + \int_0^t dt'\, a_0 = v_0 + a_0 t} \tag{12}$$

and
$$x(t) = x(0) + \int_0^t dt'\, v(t')$$

$$\boxed{x(t) = x_0 + \int_0^t dt'\,(v_0 + a_0 t') = x_0 + v_0 t + \frac{1}{2} a_0 t^2} \tag{13}$$



Here $v_0 < 0$ but $x_0 > 0$
and $a_0 > 0$

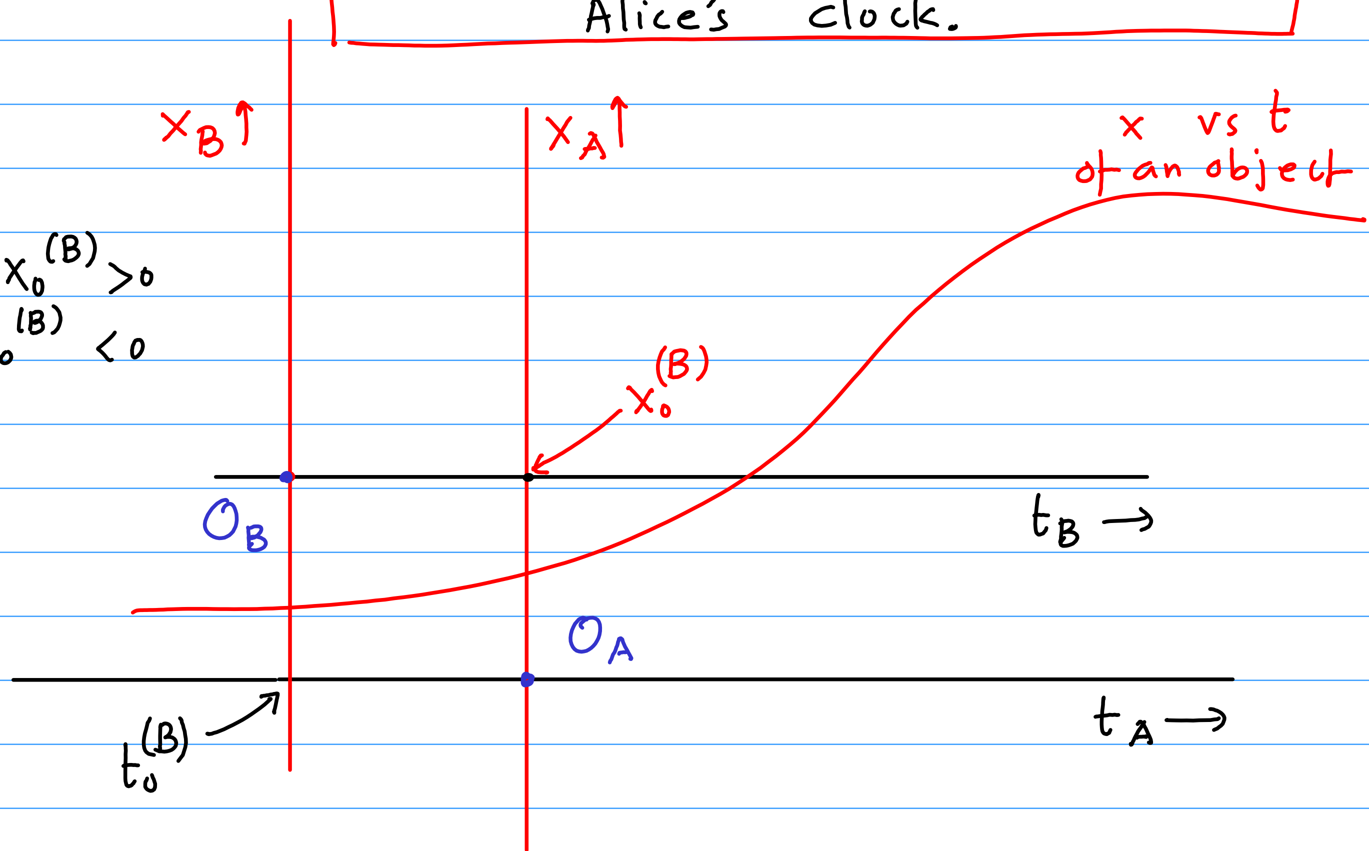
Motion under constant a results from the action of a constant net force. The simplest example is motion under gravity alone (without friction etc) near the Earth's surface.

Let's get back to Alice and Bob, who are measuring the position, velocity and acceleration of the same object, but who may have chosen different origins of position and time.

First consider the simple case when they are not moving relative to one another

Define $\boxed{\begin{aligned} x_0^{(B)} &= \text{position of Bob's origin according to Alice} \\ t_0^{(B)} &= \text{zero of Bob's time on Alice's clock.} \end{aligned}}$ (14)

Here $x_0^{(B)} > 0$ but $t_0^{(B)} < 0$



Staring at the figure it becomes clear that if Alice measures the position to be x_A then B measures it as

$$\boxed{x_B = x_A - x_0^{(B)}} \quad (15)$$

and if Alice measures the time as t_A Bob's clock reads

$$\boxed{t_B = t_A - t_0^{(B)}} \quad (16)$$

So if Alice measures the position vs time of the object to be the function

$$\boxed{x_A(t)} \quad (17)$$

Bob will measure the function

$$\boxed{x_B(t) = x_A\left(t + t_0^{(B)}\right) - x_0^{(B)}} \quad (18)$$

Will their measurements of velocity and acceleration differ?

$$\boxed{v_A(t) = \frac{dx_A}{dt}} \quad (19)$$

$$\boxed{v_B(t) = \frac{dx_B}{dt} = \frac{dx_A}{dt}\left(t + t_0^{(B)}\right) = v_A\left(t + t_0^{(B)}\right)} \quad (20)$$

$$\boxed{a_A(t) = \frac{d^2x_A}{dt^2}} \quad (21) \qquad \boxed{a_B(t) = a_A\left(t + t_0^{(B)}\right)} \quad (22)$$

As long as they shift the time to account for the difference $t_0^{(B)}$ they measure the same velocity and acceleration.

Let us make life a bit more interesting by allowing Bob to move at constant velocity w.r.t. Alice.

To simplify the situation a bit I will assume that Alice and Bob agree on the zero of time, and that their spatial origins coincide at $t=0$.

So Bob's origin in Alice's frame is

$$\boxed{x_0^{(B)}(t) = v_0^{(B)} t} \quad (23)$$

$v_0^{(B)}$ is the velocity of the origin of Bob's frame in Alice's frame.

Now consider some object which Alice measures the position of to obtain the function

$$\boxed{x_A(t)} \quad (24)$$

Clearly, Bob measures

$$\boxed{x_B(t) = x_A(t) - v_0^{(B)} t} \quad (25)$$

If Alice measures its velocity she gets

$$\boxed{v_A(t) = \frac{dx_A(t)}{dt}} \quad (26)$$

but Bob gets

$$\boxed{v_B(t) = \frac{dx_B(t)}{dt} = v_A(t) - v_0^{(B)}} \quad (27)$$

However, they get the same acceleration

$$\boxed{a_A(t) = \frac{dv_A(t)}{dt} = a_B(t) = \frac{dv_B(t)}{dt}} \quad (28)$$

Why is all this important and useful?

Because the Principle of Relativity says that the laws of Physics should be the same in both Alice's and Bob's frames.

We will understand the consequences when we start talking about forces.