Projectile Motion – Motion in 2D with constant $\vec{a}$

To describe motion in 2 dimensions we need a position vector $\vec{r}(t)$

$$\boxed{\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}} \quad (1)$$

The velocity vector $\vec{v}(t)$ is the rate of change of position with time

$$\boxed{\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{dx(t)}{dt}\,\hat{i} + \frac{dy(t)}{dt}\,\hat{j}} \quad (2)$$

$$\boxed{v_x(t) = \frac{dx}{dt}} \quad (3) \qquad \boxed{v_y(t) = \frac{dy}{dt}} \quad (4)$$

are the components of $\vec{v}$.

Note that the unit vectors, being constant and time-independent, are left alone by the $\frac{d}{dt}$, and observe that $\vec{v}$ is the vector sum of the x-velocity vector $v_x\,\hat{i}$ and the y-velocity vector $v_y\,\hat{j}$

Similarly, the acceleration vector $\vec{a}$ is (5)

$$\boxed{\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2} = \frac{dv_x}{dt}\,\hat{i} + \frac{dv_y}{dt}\,\hat{j} = a_x(t)\,\hat{i} + a_y(t)\,\hat{j}}$$

Now let us consider motion at constant velocity before we go on to the real topic.

Constant velocity $\Rightarrow$ $\boxed{\vec{v}(t) = \vec{v}_0}$ (6) (t-independent)

Clearly $\boxed{\bar{a}(t) = \frac{d\bar{v}}{dt} = 0}$ (7)

How about $\bar{r}(t)$? We know

$$\boxed{\frac{d\vec{r}(t)}{dt} = \vec{v}_0} \quad (8)$$

Break it up into components

$$\boxed{\frac{dx}{dt} = v_{0x}} \quad (9) \qquad \boxed{\frac{dy}{dt} = v_{0y}} \quad (10)$$

We already know what the answer is from one-dimensional motion, but just for fun let us look at this a different way.

$$\boxed{\frac{dx(t)}{dt} = v_{0x} = \text{constant}} \quad (11)$$

is the simplest example of a differential equation.

A quadratic eq^n $\boxed{ax^2+bx+c=0}$ (12) involves the square of the unknown x, so you know the answer must involve square roots, the inverse of the square

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}} \quad (13)$$

Similarly, the solution to a differential equation often involves integration, the inverse of differentiation.

Integrate both sides of (11)

$$\boxed{\int_0^t dt' \frac{dx(t')}{dt'} = \int_0^t dt' v_{ox} = v_{ox} t} \quad (14)$$

By the fundamental theorem of calculus the LHS is

$$\boxed{x(t) - x(0) = x(t) - x_0} \quad (15)$$

So, as before

$$x(t) = x_0 + v_{ox} t$$

Similarly $\quad y(t) = y_0 + v_{oy} t$

or $\quad \boxed{\bar{r}(t) = \bar{r}_0 + \bar{v}_0 t} \quad (16)$

Finally, we come to motion with constant acceleration.

Because gravity causes constant acceleration, this is very useful for studying the motion of thrown objects near the Earth's surface.

$$\boxed{\vec{a}(t) = \vec{a}_0 = \text{constant}} \quad (17)$$

$$\boxed{\frac{d\vec{v}(t)}{dt} = \vec{a}(t) = \vec{a}_0} \quad (18)$$

$$\Rightarrow \quad \boxed{\vec{v}(t) = \vec{v}_0 + \vec{a}_0 t} \quad (19)$$

$$v_x(t) = v_{0x} + a_{0x} t$$

$$v_y(t) = v_{0y} + a_{0y} t$$

The two components of the motion are independent. Now proceed to the next stage

$$\boxed{\frac{d\vec{r}}{dt} = \vec{v}(t) = \vec{v}_0 + \vec{a}_0 t} \quad (20)$$

Let us be bold and integrate both sides of this vector differential equation.

$$\boxed{\int_0^t dt' \, \frac{d\vec{r}(t')}{dt'} = \int_0^t dt' \left(\vec{v}_0 + \vec{a}_0 t'\right) = \vec{v}_0 t + \frac{1}{2}\vec{a}_0 t^2} \quad (21)$$

$$\text{LHS} = \boxed{\vec{r}(t) - \vec{r}(0) = \vec{r}(t) - \vec{r}_0} \quad (22)$$

$$\Rightarrow \quad \boxed{\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2} a_0 t^2} \quad (23)$$

This really stands for two eq^{n}s

$$\boxed{\begin{aligned} x(t) &= x_0 + v_{0x}\, t + \tfrac{1}{2} a_{0x} t^2 \\ y(t) &= y_0 + v_{0y}\, t + \tfrac{1}{2} a_{0y} t^2 \end{aligned}} \quad (24)$$

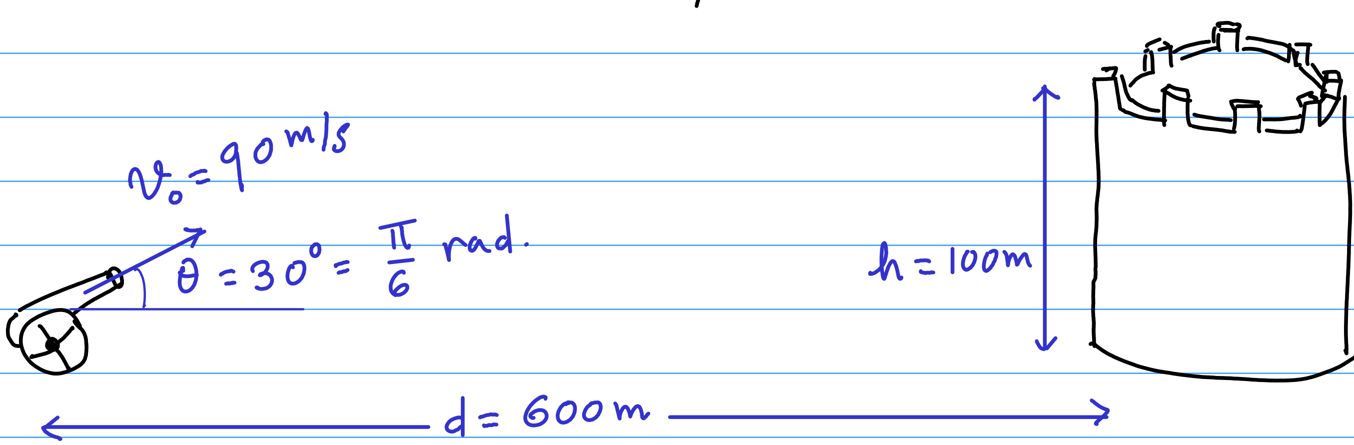
Near the Earth's surface the acceleration due to gravity is downward. The most convenient choice of coordinate axes for describing projectile motion is to choose up as the $+y$ direction.

So $\quad a_{0x} = 0 \qquad a_{0y} = -g \quad$ Minus because downwards

$$x(t) = x_0 + v_{0x}\, t$$

$$y(t) = y_0 + v_{0y}\, t - \tfrac{1}{2} g t^2$$

There are many kinds of problems one can solve. Here is a simple one.



Where does the cannonball hit on the wall?

$$v_{ox} = v_o \cos\theta = 90\,\frac{m}{s}\,\frac{\sqrt{3}}{2} = 77.9\ m/s$$

$$v_{oy} = v_o \sin\theta = 90\,\frac{m}{s}\,\frac{1}{2} = 45\ m/s$$

$$a_y = -9.8\ m/s^2$$

Start by asking how much time it takes the cannon ball to travel $\Delta x = d$. Since v_{ox} is constant

$$t = \frac{\Delta x}{v_{ox}} = \frac{600\ m}{90\sqrt{3}/2\ m/s} = \frac{40}{3\sqrt{3}}\ sec.$$

The y-position of the projectile at this time is

$$y(4\sqrt{3}\ sec) = 0 + 45\,\frac{m}{s} \times \frac{40}{3\sqrt{3}}\ sec - \frac{1}{2} \times 9.8\,\frac{m}{s^2} \times \left(\frac{40}{3\sqrt{3}}\ sec\right)^2$$

$$= 200\sqrt{3}\ m - 4.9 \times \frac{1600}{27} = 56\ m$$

The ball hits 56 m above the ground.

Now consider more difficult example. What should the ball's initial velocity be to barely clear the fortress wall if the angle is not changed?

Let's do it abstractly first and then put in numbers.

$$v_{0x} = v_0 \cos\theta \qquad v_{0y} = v_0 \sin\theta$$

time taken to have a x- displacement d is

$$t = \frac{d}{v_{0x}} = \frac{d}{v_0 \cos\theta}$$

y position at this time is

$$y(t) = 0 + \frac{v_0 \sin\theta \, d}{v_0 \cos\theta} - \frac{1}{2} g \left(\frac{d}{v_0 \cos\theta}\right)^2$$

This should be $h = 100\,m$.

or, v_0 is such that

$$d \tan\theta - \frac{1}{2} \frac{g d^2}{v_0^2 \cos^2\theta} = h$$

$$\Rightarrow \quad \frac{1}{2} \frac{g d^2}{v_0^2 \cos^2\theta} = d \tan\theta - h$$

Cross-multiply to get

$$v_0^2 = \frac{g d^2}{2 \cos^2\theta \, (d \tan\theta - h)}$$

So $$v_0 = \sqrt{\frac{g d^2}{2 \cos^2\theta \, (d \tan\theta - h)}}$$

Put in numbers to get $v_0 = 97.7\ m/s$