Rotations II - Rolling without slipping

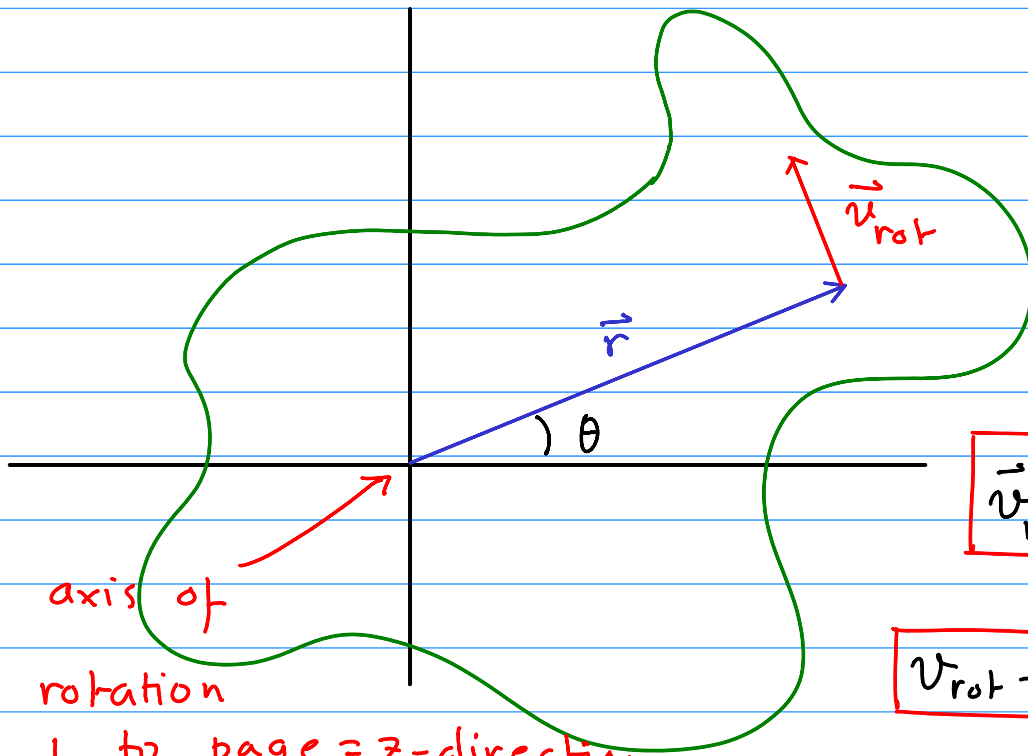
Let us collect what we know so far about rotations and the KE of rotations.

For fixed axis rotations

$\theta(t) =$ angular position ①

$\omega(t) = \frac{d\theta(t)}{dt}$ = angular velocity ②

$\alpha(t) = \frac{d\omega}{dt}$ = angular acceleration. ③



④ $\vec{v}_{rot} = \omega \hat{k} \times \vec{r}$

⑤ $v_{rot} = \omega r$

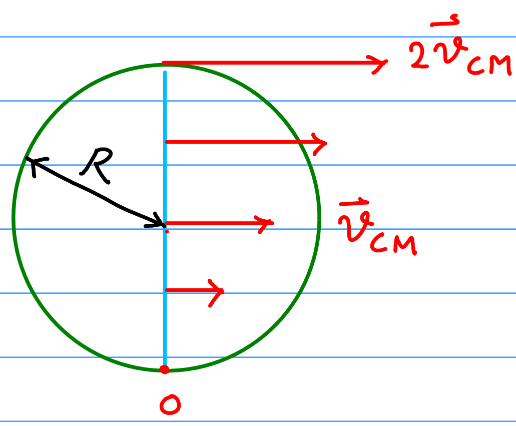
⑥ $$KE = \frac{1}{2} \sum_\alpha m_\alpha v_\alpha^2 = \frac{1}{2} \sum_\alpha m_\alpha r_\alpha^2 \omega^2 = \frac{1}{2} I \omega^2$$

where $I = \sum_{\alpha} M_\alpha r_\alpha^2$ = moment of inertia

Thin rod	Thin disk	Sphere
$I_{CM} = \frac{1}{12} M\ell^2$	$I_{CM,\perp} = \frac{1}{2} MR^2$	$I_{CM} = \frac{2}{5} MR^2$
$I_{end} = \frac{1}{3} M\ell^2$	$I_{CM,\parallel} = \frac{1}{4} MR^2$	

Rolling without slipping. When an object of circular cross-section rolls without slipping on a plane surface, the point of contact is instantaneously at rest.

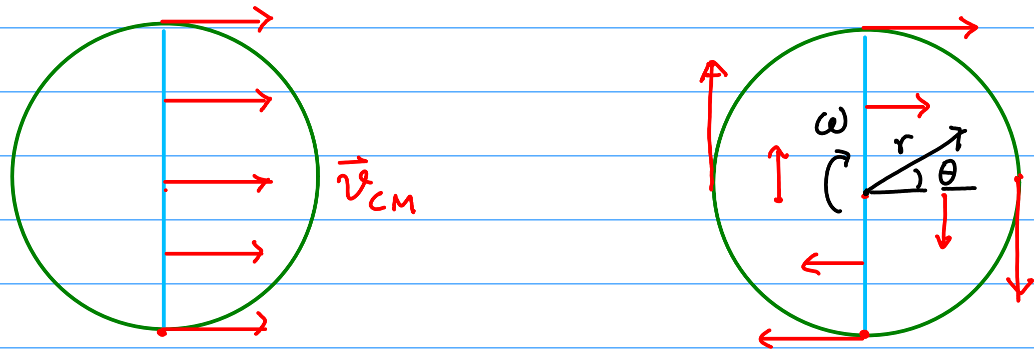
How about the other points of the body? Let us consider the case when the axis of rotation is the z-axis, $\perp$ to the page and the object is rolling in the x-direction



The velocity of the topmost point is $2\vec{v}_{CM}$.

To get a mental picture of this imagine a spool of thread. Unwind it by pulling on the string. When the string has unwound by ℓ, the CM has also moved by ℓ, so your hand is at 2ℓ. Therefore it has moved twice as fast as the CM.

Another way to think about it is to decompose the motion into a superposition of CM motion and a rotation.



What is the rotational velocity at $\vec{r}$?

$$\vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) \quad (8)$$

axis of rotation is the z-direction

$$\Rightarrow \vec{v}_{rot} = \omega\hat{k} \times \vec{r} = \omega r(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}) \quad (9)$$

$$\Rightarrow \vec{v}_{tot}(\vec{r}) = \vec{v}_{CM} + \omega r(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}) \quad (10)$$

At the point of contact $\theta = -\frac{\pi}{2}$ $r = R$

$$\vec{v}_{tot,\,bottom} = v_{CM}\,\hat{i} + \omega R(-\hat{i}) = (v_{CM} - \omega R)\,\hat{i} \qquad (11)$$

We know this must be zero, so

$$\omega = \frac{v_{CM}}{R} \qquad (12)$$

How about the topmost pt? $\theta = \pi/2$

$$\vec{v}_{tot,\,top} = v_{CM}\,\hat{i} + \omega R(\hat{i}) = 2 v_{CM}\,\hat{i} \qquad (13)$$

Now we are ready to talk about the total (translational + rotational) KE of a body. In general, it is

$$K = \int d^3\bar{r}\; \rho(\bar{r})\, \frac{1}{2}\, \vec{v}^{\,2}(\bar{r}) \qquad (14)$$

Now once again, let

$$\vec{v}(\bar{r}) = \bar{v}_{CM} + \vec{v}_{rot}(\bar{r}) \qquad (15)$$

$$K = \frac{1}{2} \int d^3\bar{r}\; \rho(\bar{r}) \left\{ v_{CM}^2 + v_{rot}^2(\bar{r}) + 2\,\vec{v}_{CM} \cdot \bar{v}_{rot} \right\} \qquad (16)$$

axis of rotation = z-direction

Now recall $\vec{v}_{rot}(\bar{r}) = \omega\, \hat{k} \times \vec{r}$ (17)

$$\Rightarrow \quad \vec{v}_{CM} \cdot \bar{v}_{rot} = \omega\, \vec{v}_{CM} \cdot (\hat{k} \times \vec{r}) \qquad (18)$$

Use the cyclic triple-product identity

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \quad (19)$$

$$\vec{v}_{CM}\cdot(\hat{k}\times\vec{r}) = \vec{r}\cdot(\vec{v}_{CM}\times\hat{k}) \quad (20)$$

$$K = \frac{1}{2}\int d^3\vec{r}\;\rho(\vec{r})\left\{\vec{v}_{CM}^{\,2} + \omega^2(\hat{k}\times\vec{r})^2 + \vec{r}\cdot\underbrace{(\vec{v}_{CM}\times\hat{k})}_{\text{indep of }\vec{r}}\right\} \quad (21)$$

The last term vanishes because

$$\int d^3\vec{r}\;\rho(\vec{r})\,\vec{r} = M\vec{R}_{CM} = 0 \quad (22)$$

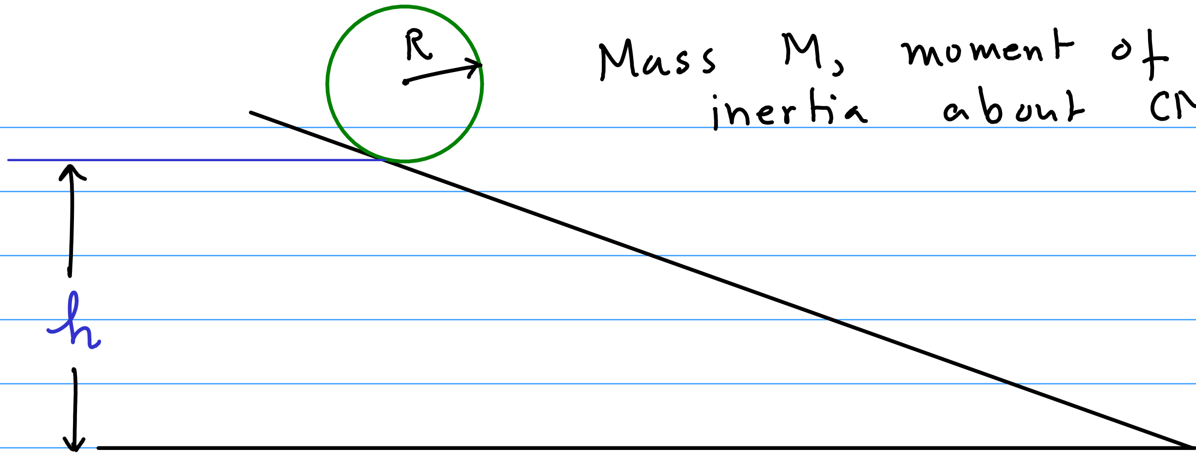
because we have chosen our origin to be the CM.

$$K = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\,\omega^2 \quad (23)$$

Now, for rolling without slipping, use (12)

$$\omega = \frac{v_{CM}}{R} \quad (24)$$

$$K = \frac{1}{2}v_{CM}^2\left\{M + \frac{I_{CM}}{R^2}\right\} \quad (25)$$



Mass M, moment of inertia about CM, I_{CM}

We know that friction exists, but because the point of contact does not move instantaneously, the work done by friction = 0

$$W_{nc} = 0 \Rightarrow \Delta E_{mech} = 0 \quad (26)$$

Initially it is at rest at a height h.

$$E_{mech,i} = E_{mech,f} \quad (27)$$

$$E_{mech,i} = Mgh \quad (28)$$

$$E_{mech,f} = K_f = \frac{1}{2} v_{CM,f}^2 \left[M + \frac{I_{CM}}{R^2} \right] \quad (29)$$

So

$$Mgh = \frac{1}{2} v_{CM,f}^2 \left[M + \frac{I_{CM}}{R^2} \right] \quad (30)$$

Suppose we had a solid cylinder

$$I_{CM} = \frac{1}{2} MR^2 \quad (31)$$

$$\Rightarrow Mgh = \frac{1}{2} v_{CM,f}^2 M \left[1 + \frac{1}{2} \right] = \frac{3}{4} M v_{CM,f}^2$$

$$v_{CM,f} = \sqrt{\frac{4gh}{3}}$$ (32) Solid cylinder

Compare to $\sqrt{2gh}$ (33) for a sliding block

For a solid sphere of uniform density

$$I_{CM} = \frac{2}{5} MR^2$$ (34)

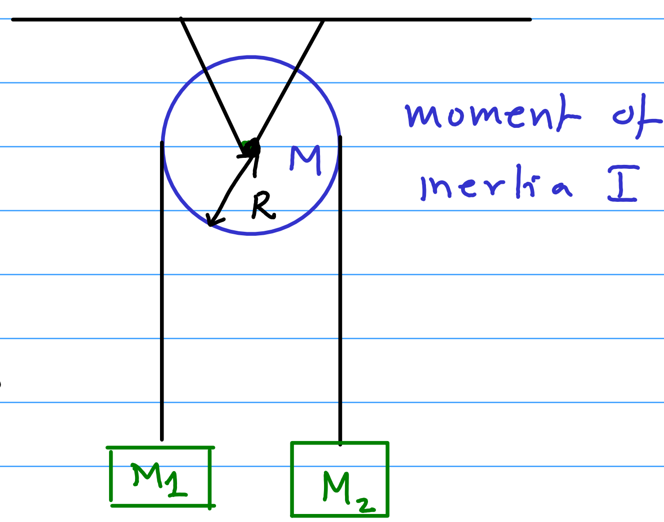
$$\Rightarrow \quad Mgh = \frac{1}{2} M v_{CM,f}^2 \left[1 + \frac{2}{5}\right] = \frac{7}{10} M v_{CM,f}^2$$

$$v_{CM,f} = \sqrt{\frac{10gh}{7}}$$ (35) Solid sphere

Let's do some more examples:

Example 1 Atwood's machine

We have analysed this problem earlier, but we considered the pulley to be massless. Now we are given that it has mass M, radius R and moment of inertia I



The masses are released from rest. Assume that $M_2 > M_1$. Clearly M_2 goes down and M_1 goes up. (36)

Because the string is inextensible, they have the same speed (though their velocities are opposite).

What about the pulley? The pulley rolls without slipping on the rope!

So $$\omega = \frac{v}{R}$$ (37)

Choose the initial height of $M_1, M_2 = 0$.
Final heights

$$h_2 = -h$$ (38) (goes down)

$$h_1 = h$$ (39) (goes up)

$$W_{nc} = 0 \Rightarrow E_{mech,i} = E_{mech,f}$$ (40)

$$E_{mech,i} = 0$$ (41)

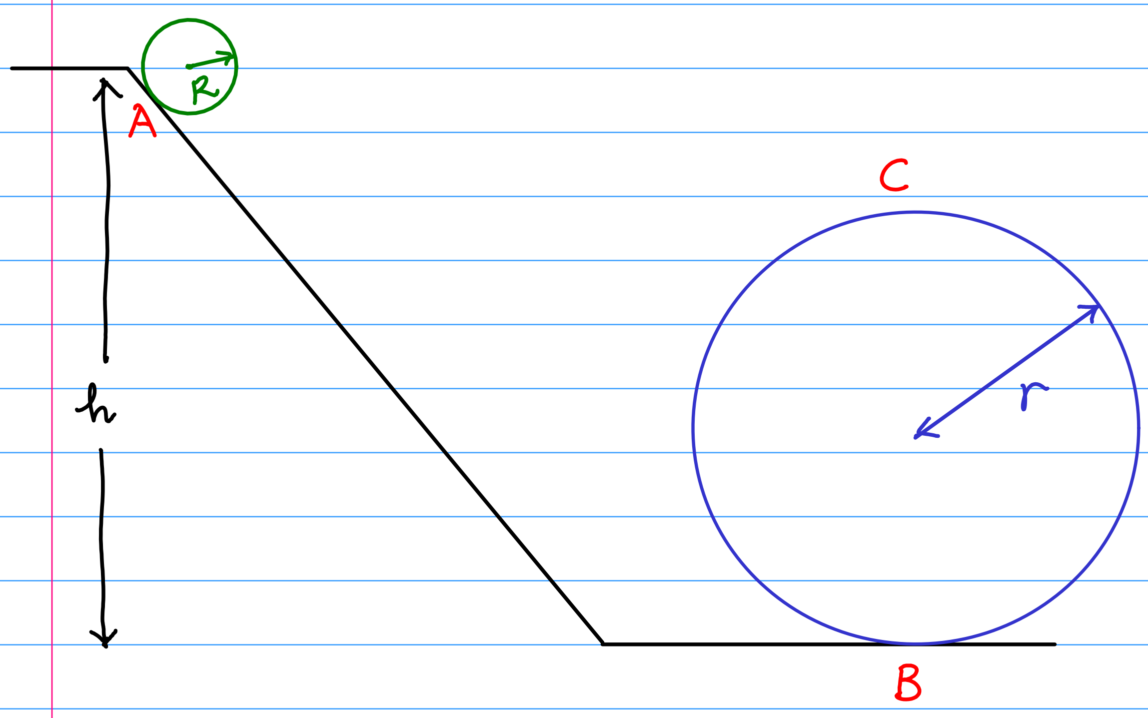
$$E_{mech,f} = M_1 g h - M_2 g h + \frac{1}{2}(M_1 + M_2) v_f^2 + \frac{1}{2} I \omega_f^2$$ (42)

or $$0 = (M_1 - M_2) g h + \frac{1}{2}\left(M_1 + M_2 + \frac{I}{R^2}\right) v_f^2$$

$$\Rightarrow \quad v_f = \sqrt{\frac{2gh(M_2 - M_1)}{M_1 + M_2 + I/R^2}} \quad (43)$$

Example 2: The loop-de-loop revisited.

Now the object rolls without slipping throughout. What is the minimum height such that the object maintains contact with the loop?



Since the object rolls without slipping

$$W_{nc} = 0 \quad \Rightarrow \quad E_{mech,A} = E_{mech,B} = E_{mech,C} \quad (44)$$

What is the speed at B? Let it be v_B. (45)

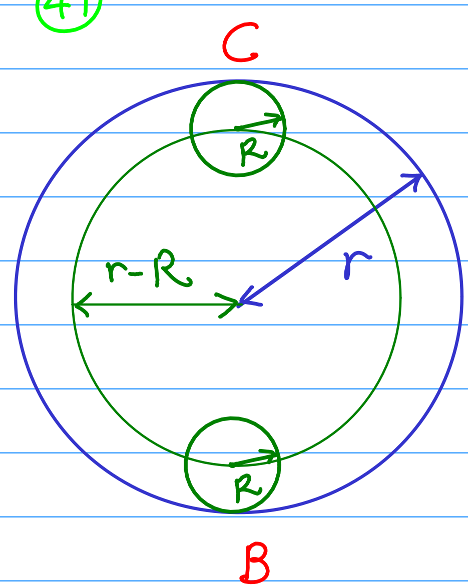
The CM of the object at B is $h_B = R$ (46)

So $E_{mech,A} = E_{mech,B}$

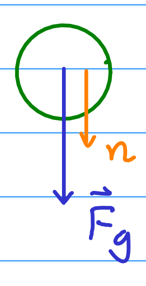
or $$Mgh = MgR + \frac{1}{2}Mv_B^2 + \frac{1}{2}I\omega^2$$
$$= MgR + \frac{1}{2}Mv_B^2 + \frac{1}{2}I\frac{v_B^2}{R^2}$$ (47)

$$v_B^2 = \frac{2Mg(h-R)}{M + I/R^2}$$ (48)

The object's CM makes a circle of radius $r-R$ inside the loop-de-loop (49)



At C the FBD is



and $$\vec{a} = -\frac{v_c^2}{r-R}\hat{j}$$ (50)

$$\Rightarrow \quad -Mg - n = -\frac{Mv_c^2}{r-R} \quad (51) \qquad n_{min} = 0 \quad (52)$$

$$v_{c,min}^2 = g(r-R) \quad (53)$$

Now

$$E_{mech,c} = \frac{1}{2}Mv_c^2 + \frac{1}{2}I\omega_c^2 + \underbrace{Mg(2r-R)}_{h_c}$$

$$= \frac{1}{2}v_c^2\left(M + \frac{I}{R^2}\right) + Mg(2r-R) \quad (54)$$

$$E_{mech,A} = E_{mech,c}$$

$$\Rightarrow \quad Mgh_{min} = \frac{1}{2}g(r-R)\left(M + \frac{I}{R^2}\right) + Mg(2r-R)$$

$$\Rightarrow \quad h_{min} = \frac{1}{2}(r-R)\left(1 + \frac{I}{MR^2}\right) + 2r - R \quad (55)$$

Note that when we had the frictionless block sliding down the slope into the loop-de-loop, the minimum height was

$$\frac{5}{2}r \quad (56)$$

We can recover this limit by setting $R \to 0$, $\frac{I}{R^2} \to 0$

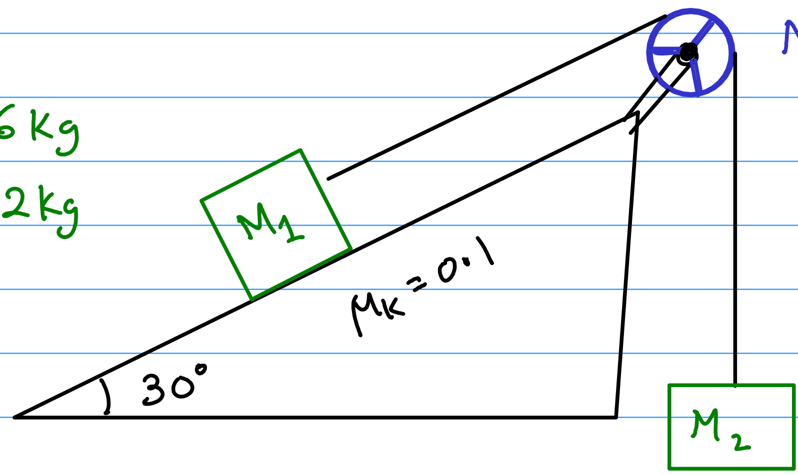
Example 3 : Pulleys and planes with friction

$M_1 = 6\,kg$

$M_2 = 2\,kg$

$M = 2\,kg \quad R = 0.2\,m.$

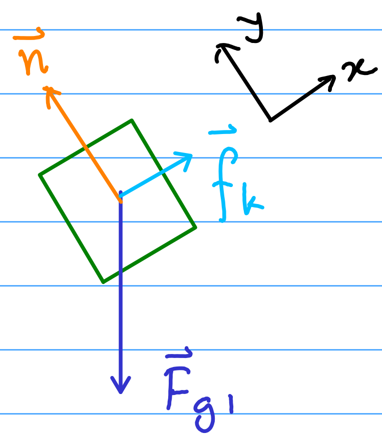
$I = MR^2 = 0.08\,kg\,m^2$



I am assuming that the pulley is a thin cylindrical shell $\boxed{I = MR^2}$ (57)

If the masses start from rest and M_1 moves down the slope by 1 m what is the final speed of M_1, M_2 and the final angular speed of the pulley?

$$\boxed{W_{nc} = E_{mech,f} - E_{mech,i}} \quad (58)$$



$$\boxed{a_y = 0 \Rightarrow F_{y,tot} = 0} \quad (59)$$

$$\Rightarrow n - M_1 g \cos 30° = 0$$

$$\boxed{n = M_1 g \cos 30°} \quad (60)$$

$$\boxed{f_k = \mu_k M_1 g \cos 30°} \quad (61)$$

$$= 0.1 \times 6\,kg \times 9.8\,\frac{m}{s^2} \times 0.866$$

$$\boxed{= 5.1\,N} \quad (62)$$

Let the initial heights be considered zero.

$h_{i1} = h_{i2} = 0$ (63)

$h_{f1} = -0.5\,m$ (which is $1m \times \sin 30°$) (64)

$h_{f2} = +1m$

$v_1 = v_2 = \omega R$ (65) Rolling w/o slipping

So $E_{mech,f} = M_1 g h_{f1} + M_2 g h_{f2} + \frac{1}{2}(M_1+M_2)v_f^2 + \frac{1}{2} I \left(\frac{v}{R}\right)^2$ (66)

$= 6 \times 9.8 \left(\frac{-1}{2}\right) + 2 \times 9.8 \times 1 + \frac{1}{2}(6+2) v_f^2 + \frac{1}{2}(0.08)\frac{v_f^2}{0.04}$

$= -9.8\,J + 5 v_f^2$ (67)

So $W_{nc} = -f_k \Delta x = -5.1\,J$ (68) $= -9.8\,J + 5 v_f^2 - \underbrace{E_{mech,i}}_{0}$

$\Rightarrow 5 v_f^2 = 4.7\,J$ $v_f = 0.97\,m/s$ (69)

and $\omega_f = \frac{v_f}{R} = 4.85$ rads/s (70)