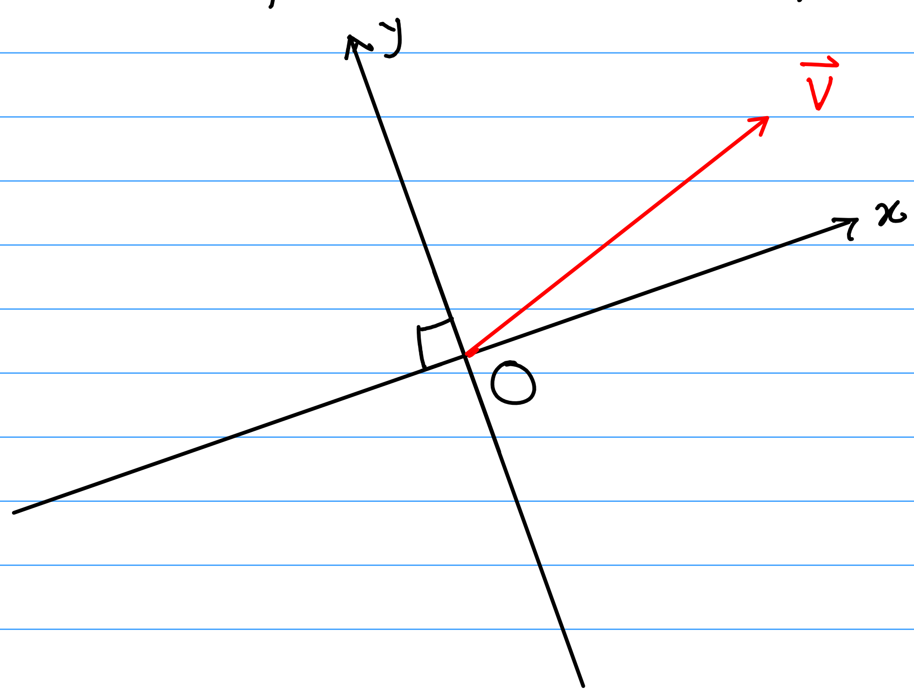
Vectors

Many physical quantities are vectors. The simplest definition of a vector is a physical quantity that has magnitude and direction.

A more precise definition involves the "transformation" between the numbers used to define a vector between Alice and Bob. More on this later.

Take the simplest case when one needs vectors to define position: 2 Dimensions. So we are talking about positions, velocities etc on a plane.

The 1st thing, as usual, is to pick an origin O and the (mutually perpendicular) directions of the x and y axes

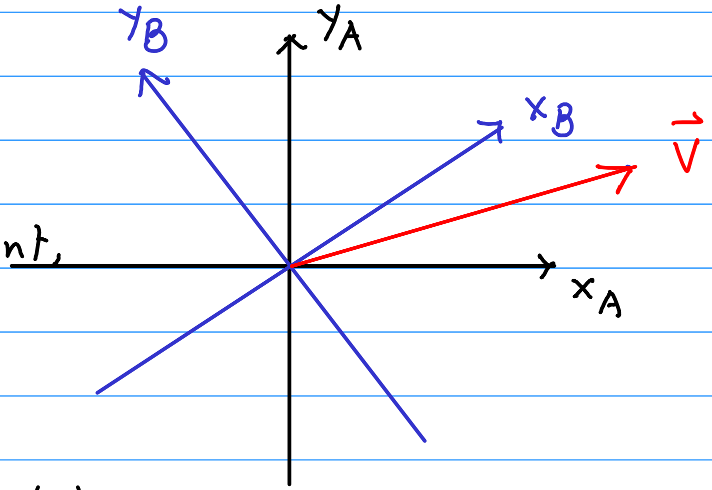


Clearly, Alice and Bob can choose different origins, and their axes may be defined differently. Let us initially work in a particular coordinate frame, Alice's.

Before talking about vectors, it is useful to talk about scalars. A scalar is a quantity which both Alice and Bob will measure to have the same value, regardless of how their coordinate axes differ. Examples of scalars are the mass of an object, the temperature at a particular point, or the kinetic energy of an object.

The same physical vector, on the other hand, will be described differently by Alice and Bob

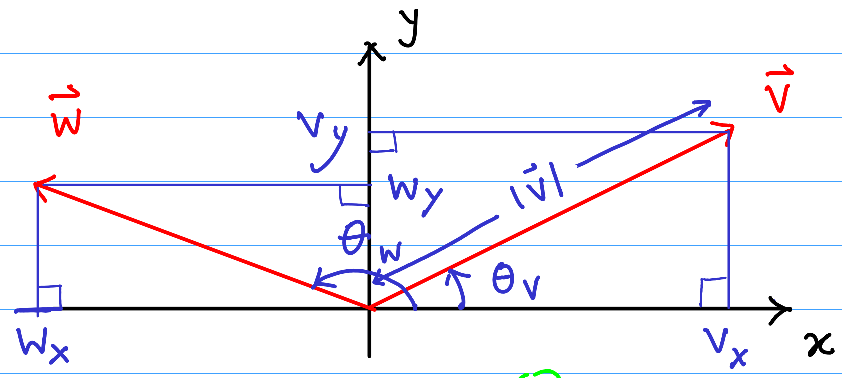
In the figure on the right, Alice sees the vector as lying in the 1st quadrant, while Bob sees it lying in the 4th.



To be more precise, let us discuss how to describe vectors.

There are two standard ways to describe a vector $\vec{V}$.

In the component form we find the projection of $\vec{v}$ on the two coordinate axes



Then $\boxed{\vec{v} = (v_x, v_y)}$ ① is defined as the ordered pair of numbers. Note that either or both of v_x, v_y can be negative

The second way is to look at the magnitude of the vector $|\vec{v}|$ and the angle θ_v ② makes with the positive x-axis. $\boxed{0 \le \theta_v < 2\pi}$

So $\boxed{\vec{v} = |\vec{v}|; \theta_v}$ ③ and $\vec{w} = |\vec{w}|, \theta_w$

Given $|\vec{v}|, \theta_v$ one can find the components by trigonometry. The vector $\vec{v}$, the x-axis and the perpendicular dropped on to it form a right-angled triangle.

So $\boxed{\sin\theta_v = \frac{v_y}{|\vec{v}|}}$ ④ $\boxed{\cos\theta_v = \frac{v_x}{|\vec{v}|}}$ ⑤

$$\Rightarrow \boxed{v_x = |\vec{v}|\cos\theta_v \qquad v_y = |\vec{v}|\sin\theta_v}$$ ⑥

How about going the other way? By the Pythagorean theorem

$$\boxed{|\vec{v}| = \sqrt{v_x^2 + v_y^2}} \quad (7) \qquad \text{and} \qquad \boxed{\theta_v = \tan^{-1}\left(\frac{v_y}{v_x}\right)} \quad (8)$$

One has to be a bit careful with the formula for θ_v if one or both of v_x, v_y are negative.

$$\boxed{\begin{array}{llll} \text{If} & v_x < 0 \text{ but } v_y > 0 & \text{then} & \theta_v = \pi - \tan^{-1}\dfrac{v_y}{|v_x|} \\ \text{If} & v_y < 0 \text{ but } v_x > 0 & \text{then} & \theta_v = 2\pi - \tan^{-1}\dfrac{|v_y|}{v_x} \\ \text{If} & v_x, v_y < 0 & \text{then} & \theta_v = \pi + \tan^{-1}\dfrac{|v_y|}{|v_x|} \end{array}} \quad (9)$$

There are several important operations one can perform on vectors. Two elementary ones are

(i) Multiply by a scalar to obtain a rescaled vector.

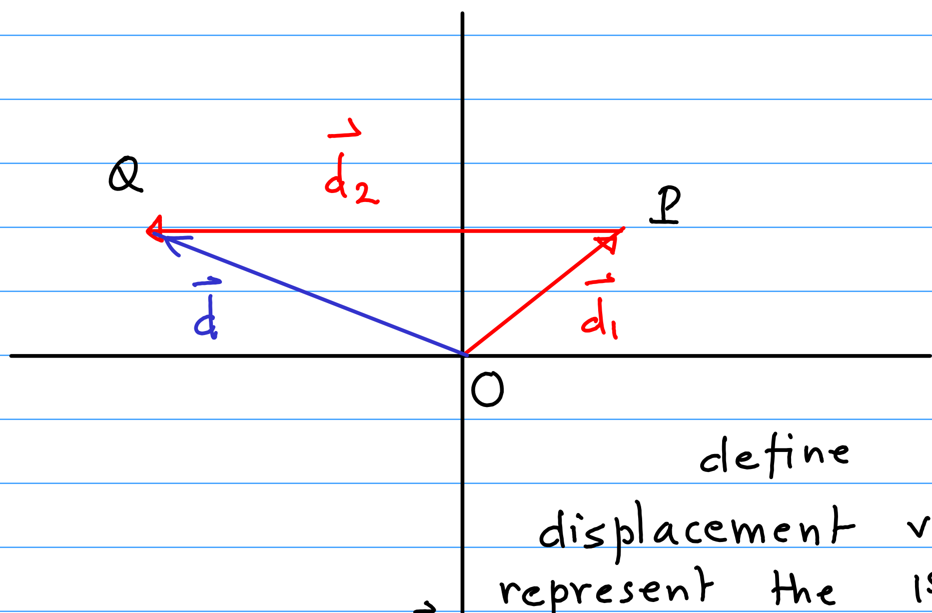
(ii) If two vectors have the same physical dimensions one can add them to obtain another vector

An example of (i) is the momentum vector $\vec{p}$ which is the mass times the velocity

$$\boxed{\vec{p} = m\vec{v}} \quad (10)$$

Consider how to add vectors. Say I start from O and travel Northeast for 5 meters to point P. Then I go due West for 10 meters to Q. How far am I from O?

Choose a set of coordinate axes centered at O with due East being the positive x-direction and due North the + y-direction.



define the displacement vector $\vec{d}_1$ to represent the 1st part of the journey. $\vec{d}_1$ has a magnitude of 5 m and makes an angle $\frac{\pi}{4}$ (45°) with + x

$\vec{d}_2$ represents the 2nd stage, with a length of 10 m and makes an angle π (180°) with + x.

Geometrically, the sum of two vectors $\vec{d}_1$, $\vec{d}_2$ obtained by placing the tail of $\vec{d}_2$ at the tip of $\vec{d}_1$ and drawing the resultant vector $\vec{d}$ from the tail of $\vec{d}_1$ to the tip of $\vec{d}_2$.

Adding vectors is much easier to do in component form.

(11)
$$\boxed{\vec{d}_1 = \left(\frac{5}{\sqrt{2}}\,m,\ \frac{5}{\sqrt{2}}\,m\right)}$$

(12)
$$\boxed{\vec{d}_2 = (-10\,m,\ 0)}$$

d_{1x} d_{1y} d_{2x} d_{2y}

Their sum is

(13)
$$\boxed{d_x = d_{1x} + d_{2x}}$$

(14)
$$\boxed{d_y = d_{1y} + d_{2y}}$$

(15)
$$\vec{d} = \left(\left(-10 + \frac{5}{\sqrt{2}}\right) m,\ \frac{5}{\sqrt{2}}\,m\right) = \boxed{(-6.446\,m,\ 3.535\,m)}$$

(16)
$$\text{distance from origin} = \boxed{\sqrt{d_x^2 + d_y^2} = 7.352\,m}$$

Subtracting vectors is just as easy. All we do is say

(17)
$$\boxed{\vec{V}_1 - \vec{V}_2 = \vec{V}_1 + (-\vec{V}_2)}$$

$-\vec{V}_2$ is the vector $\vec{V}_2$ flipped to point in the opposite direction

(18)
$$\text{If} \quad \boxed{\vec{V}_1 = (V_{1x}, V_{1y}) \qquad \vec{V}_2 = (V_{2x}, V_{2y})}$$

(19)
$$\text{then} \quad \boxed{\vec{V}_1 - \vec{V}_2 = (V_{1x} - V_{2x},\ V_{1y} - V_{2y})}$$

Cleary, we can add as many vectors as we want.

Once we can multiply vectors by scalars and also add and subtract vectors, we can define the useful notion of unit vectors 20

$\hat{i}$ = dimensionless unit vector in $+x$ direction
$\hat{j}$ = dimensionless unit vector in $+y$ direction

This gives us a new way to express a vector $\vec{v} = (v_x, v_y)$

$$\boxed{\vec{v} = v_x \hat{i} + v_y \hat{j}}$$ 21

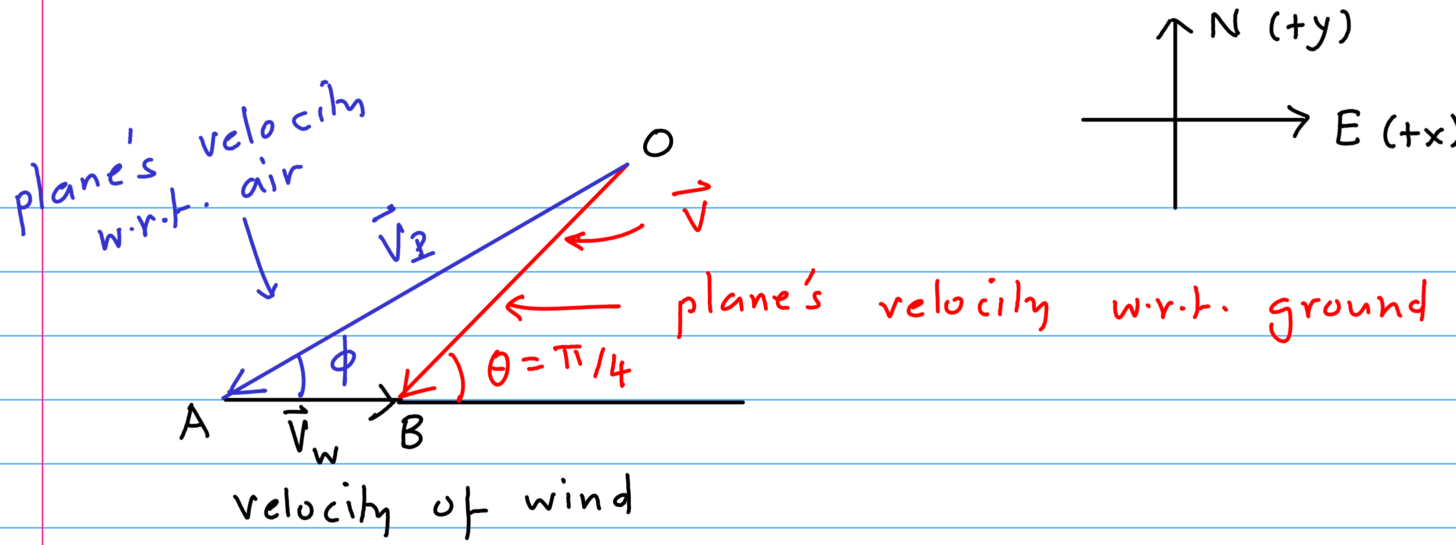
If we work in 3 dimensions we have to introduce a third unit vector $\hat{k}$ in the $+z$ direction

Adding and subtracting vectors is crucial in problems of relative motion.

For example, a plane travels at an airspeed of 500 km/hr. Suppose it wants to go from Lexington to Houston TX, which is 1500 km away Southwest.

If there is no wind the pilot points the plane SW and it takes 3 hrs to get there.

Suppose there is a wind of 100 km/hr from due west. What direction does the plane have to point and how long does it take?



Clearly $\boxed{\vec{V} = \vec{V}_p + \vec{V}_w}$ (22)

$\boxed{|\vec{V}_p| = 500 \text{ km/hr}}$ (23) $\boxed{|\vec{V}_w| = 100 \text{ km/hr}}$ (24)

and $\theta = \pi/4$ since the resultant velocity must be south west

In component form (25)

$$\boxed{\begin{aligned} V_{px} &= -500 \text{ km/hr} \cos\phi \\ V_{py} &= -500 \text{ km/hr} \sin\phi \end{aligned}}$$

$\boxed{V_{wx} = 100 \text{ km/hr} \qquad V_{wy} = 0}$ (26)

(27)

2) $$\boxed{\begin{aligned} V_x &= -500 \frac{\text{km}}{\text{hr}} \cos\phi + 100 \frac{\text{km}}{\text{hr}} = -|\vec{V}| \cos\frac{\pi}{4} = -\frac{|\vec{V}|}{\sqrt{2}} \\ V_y &= -500 \frac{\text{km}}{\text{hr}} \sin\phi = -|\vec{V}| \sin\frac{\pi}{4} = -\frac{|\vec{V}|}{\sqrt{2}} \end{aligned}}$$

$V_x = V_y \Rightarrow$

$$500 \cos\phi - 100 = 500 \sin\phi$$

$$\boxed{5\cos\phi - 1 = 5\sin\phi}$$ (28)

Square both sides

$$25\cos^2\phi - 10\cos\phi + 1 = 25\sin^2\phi$$

Use $\boxed{\sin^2\phi = 1 - \cos^2\phi}$ 29

$$25\cos^2\phi - 10\cos\phi + 1 = 25 - 25\cos^2\phi$$

or $50\cos^2\phi - 10\cos\phi - 24 = 0$

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$$\boxed{\cos\phi = \frac{10 \pm \sqrt{100 + 4\times 50\times 24}}{100} = \frac{10 \pm \sqrt{4900}}{100}}$$

$\boxed{\cos\phi = \frac{4}{5}}$ 31 or $\boxed{-\frac{3}{5}}$ 32

From the figure $\phi < \frac{\pi}{2}$ so $\cos\phi > 0$

$$\cos\phi = \frac{4}{5} \Rightarrow \boxed{\begin{aligned}\phi &= 0.6435 \text{ rad} \\ &= 36.87^\circ\end{aligned}}$$ 33

Now we can find v_x, v_y

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$$\boxed{V_x = -500\frac{km}{hr}\cdot\frac{4}{5} + 100\frac{km}{hr} = -300\frac{km}{hr} = V_y}$$

$$\boxed{|\vec{v}| = 300\frac{km}{hr}\sqrt{2} = 424\frac{km}{hr}}$$ ground speed

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It takes $\boxed{\dfrac{1500 \text{ km}}{424 \frac{km}{hr}} = 3.77 \text{ hrs}}$ to get to Houston

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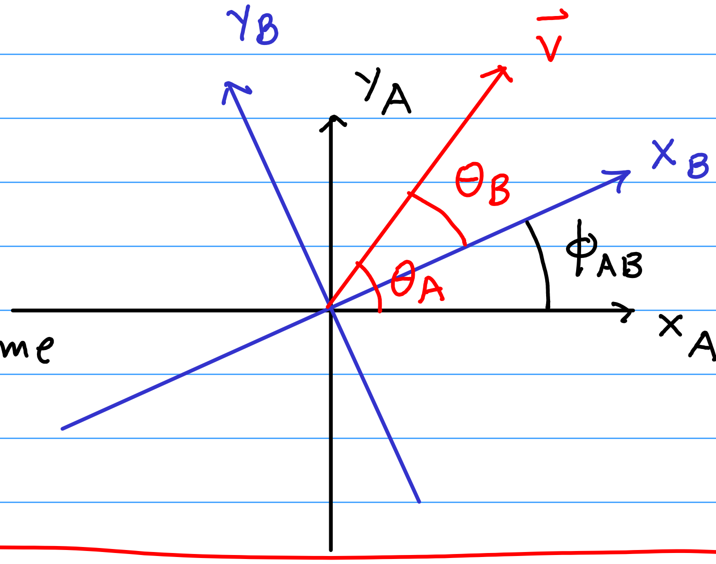
Alice, Bob, and transformations of coordinates:

Back to Alice and Bob, who have chosen different directions for their $+x$.



In Alice's frame

$$\boxed{\vec{v} = (|\vec{v}|, \theta_A)} \quad (37)$$

while in Bob's frame

$$\boxed{\vec{v} = (|\vec{v}|, \theta_B)} \quad (38)$$

Clearly if $\boxed{\phi_{AB} = \text{angle of Bob's } +x \text{ axis in Alice's frame}}$ (39)

$$\boxed{\theta_B = \theta_A - \phi_{AB}} \quad (40)$$

In components

$$\boxed{\begin{aligned} V_x^{(A)} &= |\vec{v}| \cos\theta_A & V_y^{(A)} &= |\vec{v}| \sin\theta_A \\ V_x^{(B)} &= |\vec{v}| \cos\theta_B & V_y^{(B)} &= |\vec{v}| \sin\theta_B \end{aligned}} \quad (41)$$

$$\boxed{\begin{aligned} \cos\theta_B &= \cos(\theta_A - \phi_{AB}) = \cos\theta_A \cos\phi_{AB} + \sin\theta_A \sin\phi_{AB} \\ \sin\theta_B &= \sin(\theta_A - \phi_{AB}) = \sin\theta_A \cos\phi_{AB} - \cos\theta_A \sin\phi_{AB} \end{aligned}} \quad (42)$$

So $V_x^{(B)} = \underbrace{|\vec{V}| \cos\theta_A}_{V_x^{(A)}} \cos\phi_{AB} + \underbrace{|\vec{V}| \sin\theta_A}_{V_y^{(A)}} \sin\phi_{AB}$

$$V_x^{(B)} = V_x^{(A)} \cos\phi_{AB} + V_y^{(A)} \sin\phi_{AB}$$

$$V_y^{(B)} = \underbrace{|\vec{V}| \sin\theta_A}_{V_y^{(A)}} \cos\phi_{AB} - \underbrace{|\vec{V}| \cos\theta_A}_{V_x^{(A)}} \sin\phi_{AB}$$

$$= -V_x^{(A)} \sin\phi_{AB} + V_y^{(A)} \cos\phi_{AB}$$

To repeat

$$\boxed{\begin{aligned} V_x^{(B)} &= V_x^{(A)} \cos\phi_{AB} + V_y^{(A)} \sin\phi_{AB} \\ V_y^{(B)} &= -V_x^{(A)} \sin\phi_{AB} + V_y^{(A)} \cos\phi_{AB} \end{aligned}} \quad (43)$$

Why is this a big deal? Because this same "translation" works for any two-dimensional vector.

A useful way to write this is in terms of a rotation matrix

$$\boxed{\begin{pmatrix} V_x^{(B)} \\ V_y^{(B)} \end{pmatrix} = \underbrace{\begin{bmatrix} \cos\phi_{AB} & \sin\phi_{AB} \\ -\sin\phi_{AB} & \cos\phi_{AB} \end{bmatrix}}_{\text{Rotation matrix } \mathbb{R}(\phi_{AB})} \begin{pmatrix} V_x^{(A)} \\ V_y^{(A)} \end{pmatrix}} \quad (44)$$

Mathematically, any object whose components transform like this under rotations is defined as a vector.

When one goes to 3D, the kinds of rotations one can carry out become much more complicated, but given any two coordinate frames, there is always a unique 3×3 rotation matrix which transforms the components of vectors.