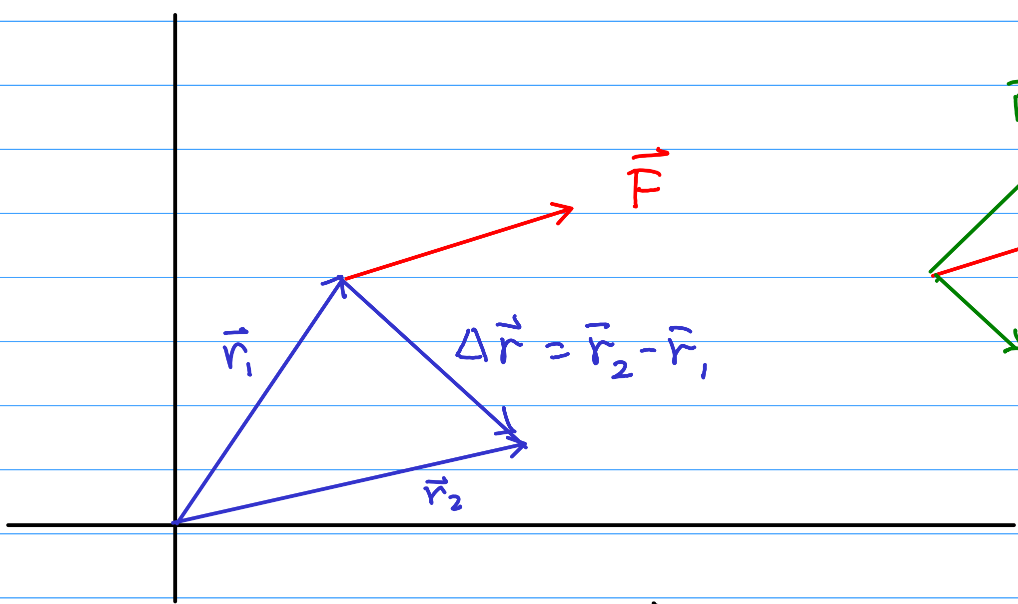
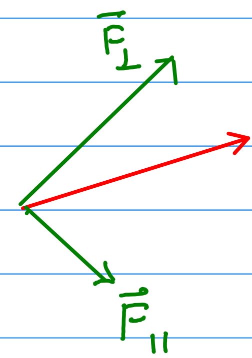
Work and Kinetic Energy

Work has a rigorous, precise, meaning in physics which is often different from the colloquial meaning.

In physics, $\boxed{\text{Work} = \text{Force} \times \text{displacement}}$ (1)
but it is a bit more complicated because both force and displacement are vectors.

Let us start with a constant force $\vec{F}$ acting on a particle which undergoes a displacement $\Delta\vec{r}$





Break up $\vec{F}$ into components parallel ($\vec{F}_\parallel$) and perpendicular ($\vec{F}_\perp$) to $\Delta\vec{r}$. Then the work done by the force is

$$\boxed{W = F_\parallel \, \Delta r} \quad (2)$$

$F_\parallel = |\vec{F}_\parallel|$

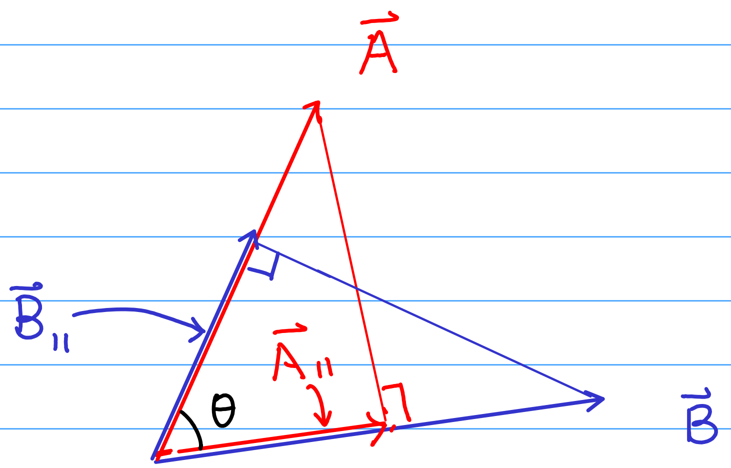
$\Delta r = |\Delta\vec{r}|$

The Dot product of two vectors

From two vectors $\vec{A}$ and $\vec{B}$ one can construct a scalar product $\vec{A}\cdot\vec{B}$ (pronounced A dot B) which is

$$\boxed{\vec{A}\cdot\vec{B} = A_{\parallel}\, B = B_{\parallel}\, A} \quad (3)$$

as shown



$\vec{B}_{\parallel}$ is the component of $\vec{B}$ parallel to $\vec{A}$

$\vec{A}_{\parallel}$ is the component of $\vec{A}$ parallel to $\vec{B}$

Clearly $$\boxed{A_{\parallel} = A\cos\theta} \quad (4)$$

So $$\boxed{\vec{A}\cdot\vec{B} = AB\cos\theta} \quad (5)$$

Similarly $$\boxed{B_{\parallel} = B\cos\theta} \quad (6)$$

So $\vec{A}\cdot\vec{B} = AB\cos\theta$ as before

The dot product is a scalar because two observers in different coordinate systems will agree on its value. It only depends on the angle between $\vec{A}$ and $\vec{B}$, and not on the angles between the vectors and the coordinate axes.

The dot product is symmetric

$$\boxed{\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}} \quad (7)$$

The reason I mention this is that there is another type of product between two vectors, the cross product, which is antisymmetric and results in another vector.

So using the dot product

$$\boxed{W = \vec{F}\cdot\Delta\vec{r}} \quad (8)$$

The unit of work = $\boxed{\text{Newton}\cdot\text{meter} = \text{Joule}}$ (9)

If one is given the magnitudes of $\vec{F}$ and $\Delta\vec{r}$ and the angle between them it is easy to find $\vec{F}\cdot\Delta\vec{r}$.

However, often one is given the various components of $\vec{F}$ and $\Delta\vec{r}$.

Let us find the dot product $\vec{A}\cdot\vec{B}$ when $\vec{A}$ and $\vec{B}$ are in component form

$$\boxed{\begin{aligned}\vec{A} &= A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \\ \vec{B} &= B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\end{aligned}} \quad (10)$$

$$\vec{A}\cdot\vec{B} = (A_x\hat{i}+A_y\hat{j}+A_z\hat{k})\cdot(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})$$
$$= A_xB_x\,\hat{i}\cdot\hat{i} + A_xB_y\,\hat{i}\cdot\hat{j} + A_xB_z\,\hat{i}\cdot\hat{k}$$
$$+ A_yB_x\,\hat{j}\cdot\hat{i} + A_yB_y\,\hat{j}\cdot\hat{j} + A_yB_z\,\hat{j}\cdot\hat{k}$$
$$+ A_zB_x\,\hat{k}\cdot\hat{i} + A_zB_y\,\hat{k}\cdot\hat{j} + A_zB_z\,\hat{k}\cdot\hat{k} \qquad (11)$$

Now we use the fact that $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular. So, $\hat{i}$ has zero component parallel to $\hat{j}$ and $\hat{k}$ etc

So $$\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0 \qquad (12)$$

On the other hand $$\hat{i}\cdot\hat{i} = |\hat{i}|\,|\hat{i}|\cos(0) = 1 \qquad (13)$$

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \qquad (14)$$

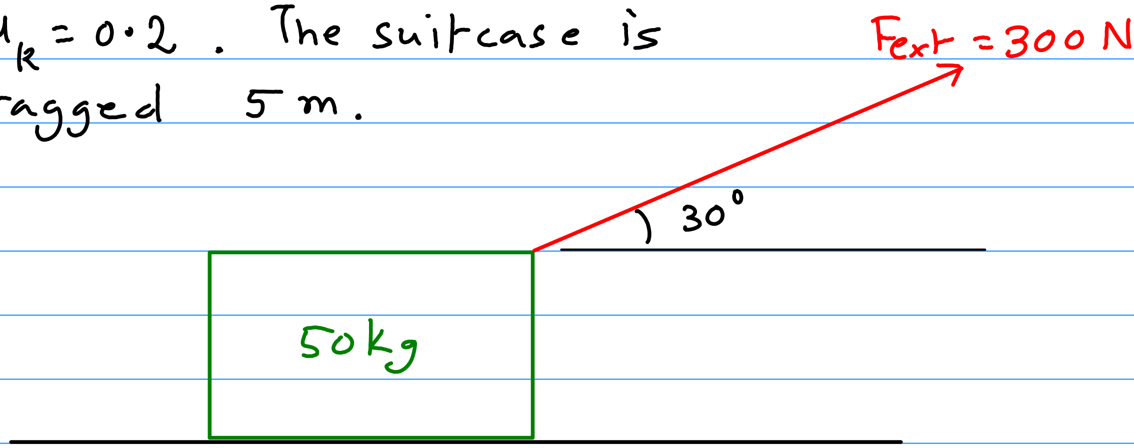
Finally $$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \qquad (15)$$

Often one talks about the dot product of a vector with itself. In this case there is a special notation

$$\vec{A}\cdot\vec{A} = \vec{A}^{\,2} \quad \text{(A vector squared)}$$
$$= A_x^2 + A_y^2 + A_z^2 \qquad (16)$$

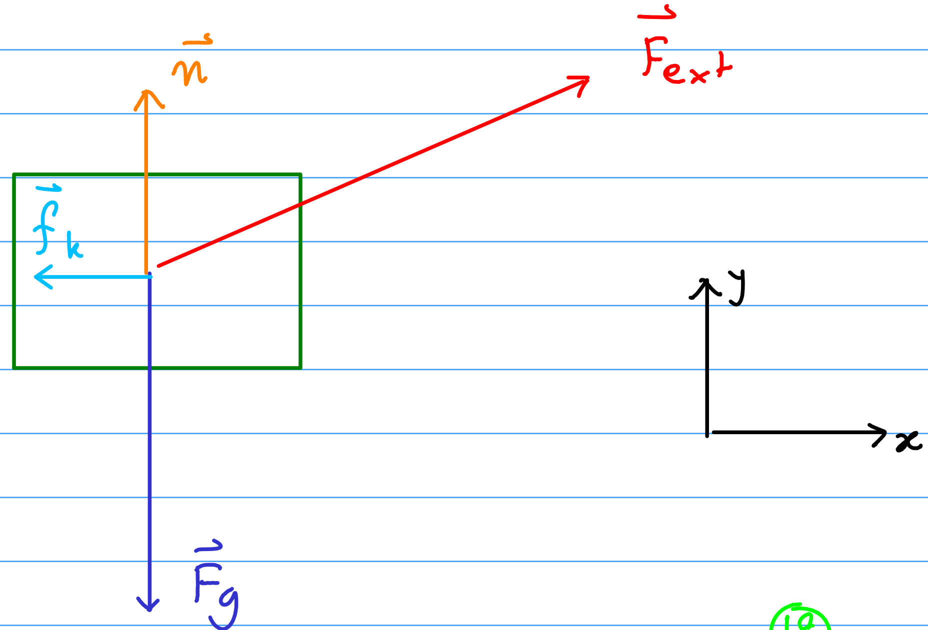
Example: The lady dragging the suitcase

$\mu_k = 0.2$. The suitcase is dragged 5 m.



What is the work done by the various forces?

FBD



(17)

$$\boxed{\vec{F}_g = -mg\,\hat{j}}$$

(18)

$$\boxed{\vec{F}_{ext} = 300\,N\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)}$$

(19)

$$\boxed{\vec{n} = n\,\hat{j}}$$

(20)

$$\boxed{\vec{f}_k = -f_k\,\hat{i}}$$

(21)

$$\boxed{\vec{F}_{tot} = \hat{i}\left(300N\,\frac{\sqrt{3}}{2} - f_k\right) + \hat{j}\left(n - mg + 150N\right)}$$

Since $a_y = 0$ $\boxed{n = mg - 150N = 490N - 150N = 340N}$ 22

$\Rightarrow$ $\boxed{f_k = \mu_k n = 0.2 \times 340N = 68N}$ 23

$F_{tot,x} = 300N \frac{\sqrt{3}}{2} - 68N = 191.8N$

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$= ma_x = 50 a_x \Rightarrow \boxed{a_x = 3.84 \, m/s^2}$

Now we want the work done by the various forces on the suitcase when

$\boxed{\Delta\vec{r} = 5m\,\hat{i}}$ 25

Since $\vec{F}_g$ and $\vec{n}$ are perpendicular to $\Delta\vec{r}$

$$\boxed{\begin{aligned} W_g &= \vec{F}_g \cdot \Delta\vec{r} = 0 \\ W_n &= \vec{n} \cdot \Delta\vec{r} = 0 \end{aligned}}$$ 26

For F_{ext} $W_{ext} = \vec{F}_{ext} \cdot \Delta\vec{r} = F_{ext} \cos 30° \, \Delta r$

$= 300N \frac{\sqrt{3}}{2} \times 5m = 1299 \, Nm =$

$\boxed{W_{ext} = 1299 J}$ 27

The abbreviation for Joules is J

How about $W_f = \vec{f}_k \cdot \Delta\vec{r}$

the angle between $\vec{f}_k$ and $\Delta\vec{r}$ is $\pi = 180°$

$\cos(\pi) = -1$

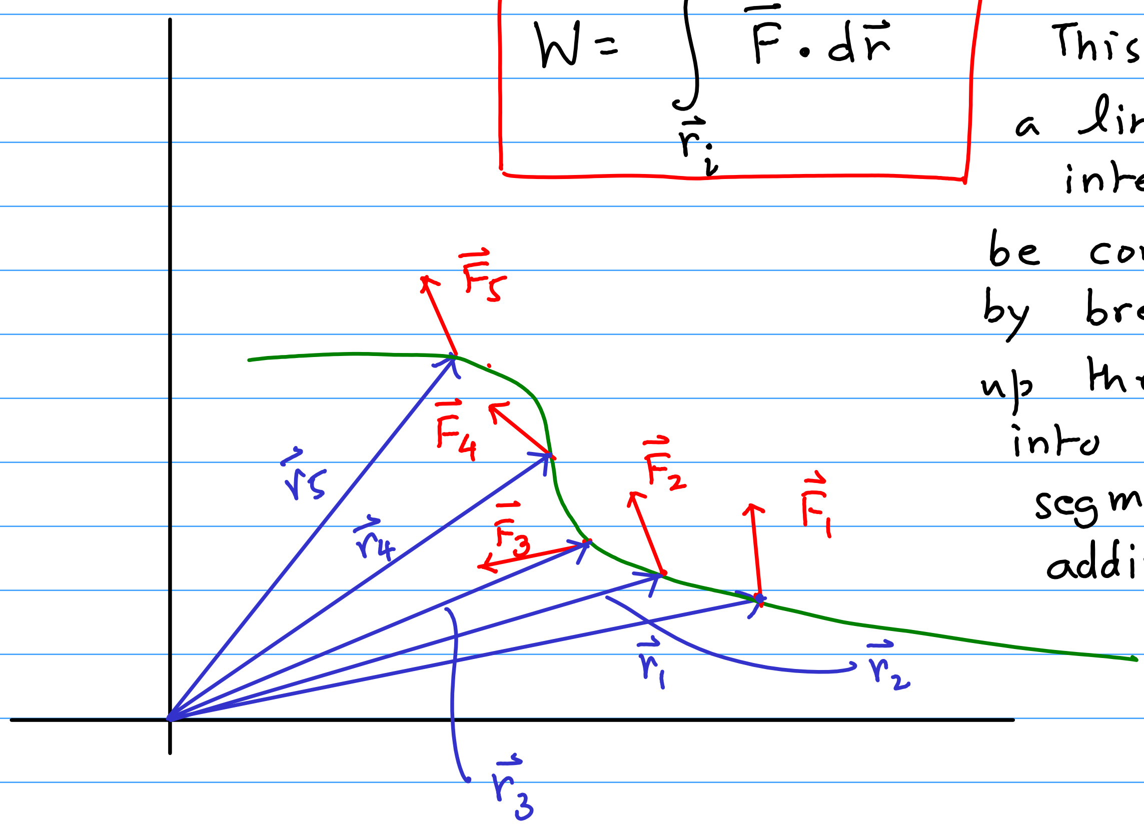
(28)
$$\boxed{W_f = -f_k \Delta r = -68N \times 5m = -340 J}$$

So Work can be negative!! This is another way in which the physics meaning differs from the colloquial one.

Now let us talk about forces that vary with position.

$$\boxed{\vec{F} \equiv \vec{F}(\vec{r})} \quad (29)$$

$$\boxed{W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}} \quad (30)$$

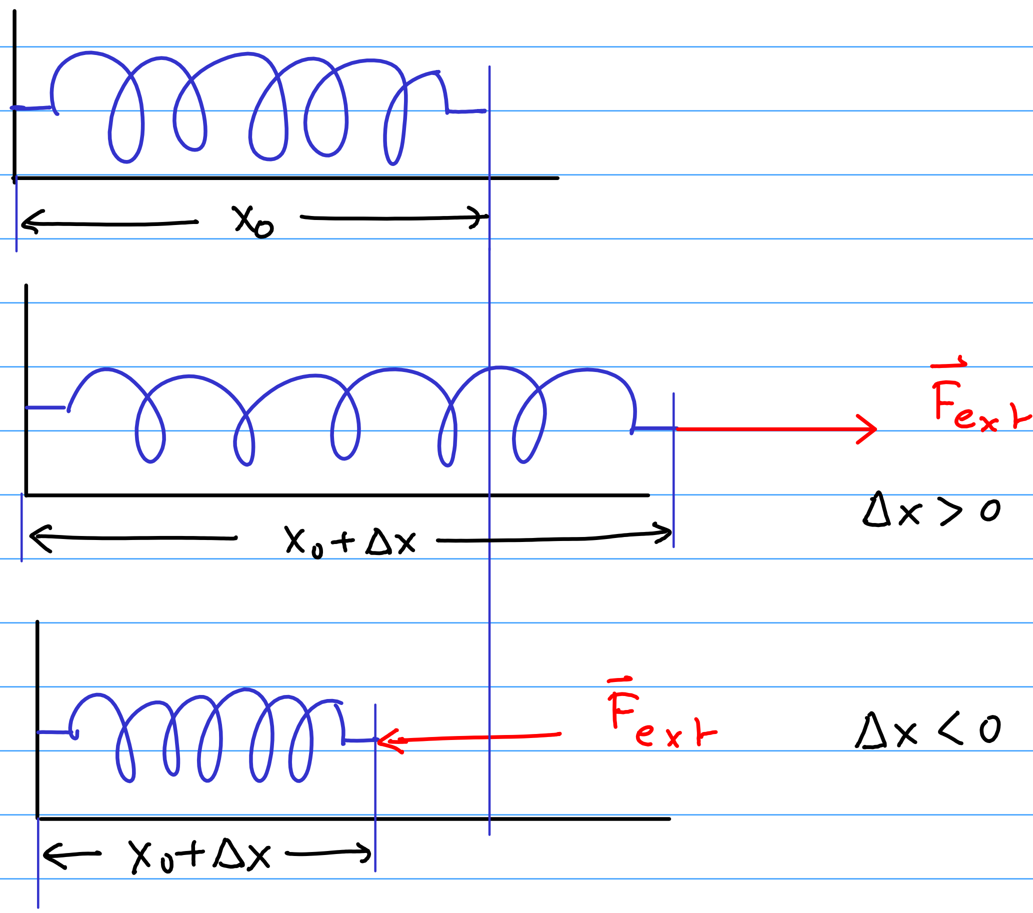


This is a line integral, to be computed by breaking up the curve into small segments and adding up $\vec{F} \cdot \Delta\vec{r}$

Let us take a one-dimensional example of a position-dependent force which we will use extensively in the future.

The linear or Harmonic spring

A spring has an equilibrium length x_0. When a force is applied on it it either stretches or compresses, depending on the direction of the applied force. The displacement of the end of the spring is Δx



For any applied force there is a particular Δx at which the spring is at a new equilibrium. This occurs because the spring produces a restoring force

$$\boxed{F_s = -k\,\Delta x} \quad (31)$$

Hooke's Law

where k is called the Force constant of the spring, or sometimes the spring constant.

The units of k are Newtons/meter.

A soft spring (like a bedspring) would have

$$\boxed{k \simeq 10^3 - 10^4 \ \mathrm{N/m}} \quad (32)$$

while a hard spring would have

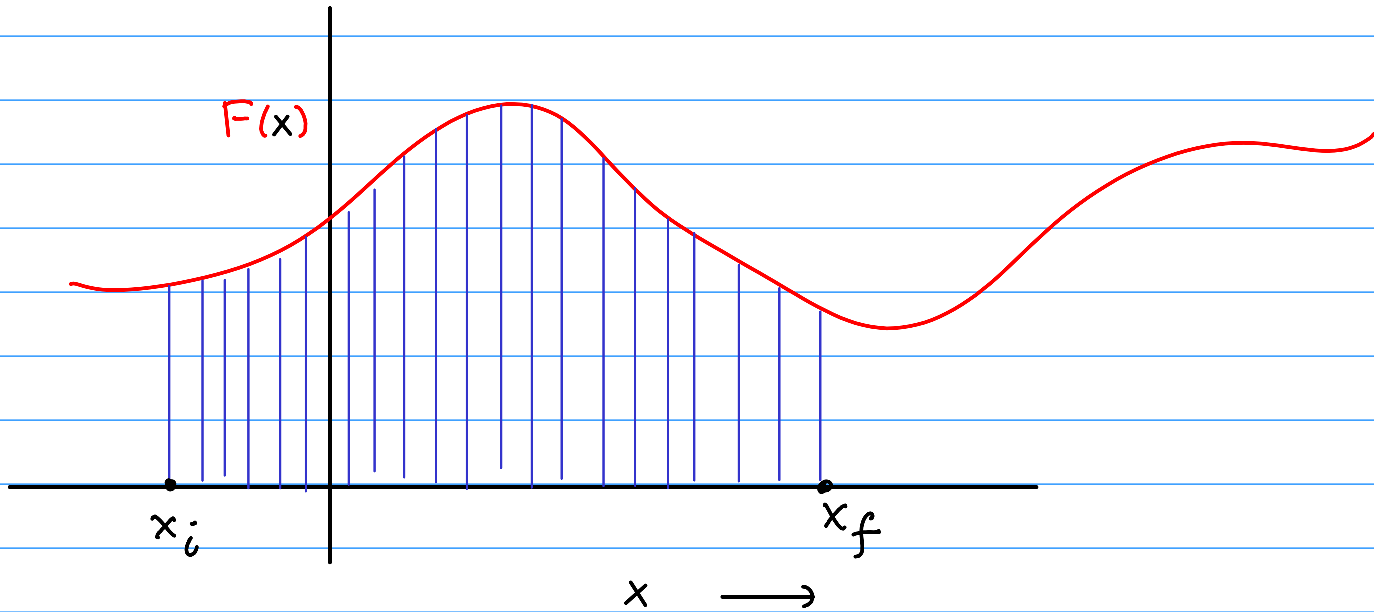
$$\boxed{k \simeq 10^5 - 10^6 \ \mathrm{N/m}} \quad (33)$$

Now, the restoring force acts on the object that is attached to the spring.

So, the work done by the spring on the object is

$$\boxed{W(\Delta x) = \int_0^{\Delta x} dx'\, F_s(x') = -\int_0^{\Delta x} k x'\, dx' = -\frac{1}{2} k (\Delta x)^2} \quad (34)$$

In 1 dimension the work done by a force that depends on position can be thought of as the area under the $F(x)$ vs x curve



Back to higher dimensions. Consider the gravitational field of the Earth on a satellite. We will learn this in detail later but the force is

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$$\boxed{\vec{F}_g = -\frac{G M_E m}{r^2}\hat{e}_r}$$

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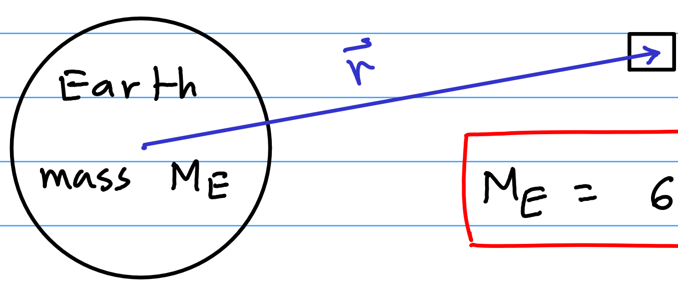
$\hat{e}_r$ = radial outward unit vector

$$\boxed{G = 6.67 \times 10^{-11}\ Nm^2/kg^2 = \text{Newton's constant}}$$ 37



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Satellite

mass $\boxed{m = 100\,kg}$

$$\boxed{M_E = 6 \times 10^{24}\ kg}$$ 39

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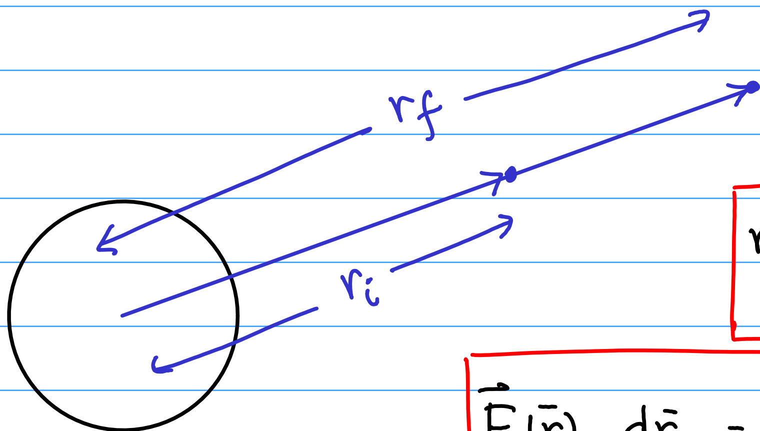
Suppose the satellite moves radially 41
outward from $\boxed{r_i = 14000\,\text{km}}$ to $\boxed{r_f = 20\,000\,\text{km}}$
All distances are measured from the Earth's center.



$$\boxed{d\vec{r} = \hat{e}_r \, dr}$$ 42

because the displacement is radially outward

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$$\boxed{\begin{aligned} r_i &= 14\times10^3\,\text{km} = 14\times10^6\,\text{m} \\ r_f &= 20\times10^6\,\text{m} \end{aligned}}$$

$$\boxed{\vec{F}(\vec{r})\cdot d\vec{r} = -\frac{G M_E m}{r^2}\, dr}$$ 44

$$\Rightarrow \boxed{W = -\int_{r_i}^{r_f} dr \, \frac{G M_E m}{r^2} = G M_E m \left[\frac{1}{r_f} - \frac{1}{r_i}\right]}$$

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$$6.67\times10^{-11}\,\frac{\text{N m}^2}{\text{kg}^2} \times 6\times10^{24}\,\text{kg} \times 100\,\text{kg} \left[\frac{1}{20\times10^6} - \frac{1}{14\times10^6}\right]$$

$$W_g = 6.67\times6\times10^{9}\left[\frac{1}{20} - \frac{1}{14}\right] = \boxed{-8.57\times10^{8}\ \text{J}}$$

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The Work-Energy Theorem

Energy is another word that has a rigorous, precise meaning in physics, which is not always how the word is used colloquially.

In physics, we recognize several categories of energy.

Any object has a kinetic energy, the energy associated with its motion.

$$\boxed{KE = \frac{1}{2} m \vec{v}^{\,2} = \frac{1}{2} m \left(v_x^2 + v_y^2 + v_z^2 \right)} \tag{47}$$

An object can also have various forms of potential energy. Potential energy is usually "hidden", and depends on the position of the object in relation to other objects. For example, an object held above ground level has a positive gravitational potential energy. An object held against a compressed spring has spring potential energy, etc.

Energy is a scalar quantity.

Work and energy are deeply connected. Note the dimensions of work

$$\boxed{[W] = [F][\Delta r] = M L T^{-2} \; L = M L^2 T^{-2}} \tag{48}$$

This is identical to the dimensions of KE

$$\boxed{[KE] = [m][v]^2 = M L^2 T^{-2}} \quad (49)$$

In fact, work and energy can be converted into each other.

The deepest result is the Work-Kinetic Energy Theorem.

Consider the total work done on an object. The force to use here is the total force

$$\boxed{W_{tot} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{tot} \cdot d\vec{r}} \quad (50)$$

$$\boxed{\vec{F}_{tot} = m\vec{a} = m \frac{d^2\vec{r}}{dt^2}} \quad (51)$$

By Newton's II Law

$$\boxed{W_{tot} = \int_{\vec{r}_i}^{\vec{r}_f} m \frac{d^2\vec{r}}{dt^2} \cdot d\vec{r}} \quad (52)$$

the integral is taken along some curve that follows the motion. Label the points on the curve by the time t and change the variable of integration to t

$$\boxed{W_{tot} = m \int_{t_i}^{t_f} dt \, \frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2}} \quad (53)$$

However (54)

$$\boxed{\frac{d}{dt}\left\{\frac{1}{2}\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\right\} = \frac{1}{2}\frac{d}{dt}\left(\vec{v}^{\,2}\right) = \frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}}$$

So

$$\boxed{W_{tot} = \frac{1}{2}m\int_{t_i}^{t_f} dt\,\frac{d}{dt}\left(\frac{1}{2}\vec{v}(t)^2\right)} \quad (55)$$

The integral is the inverse of the derivative

$$\boxed{W_{tot} = \frac{1}{2}m\vec{v}_f^{\,2} - \frac{1}{2}m\vec{v}_i^{\,2} = KE_f - KE_i = \Delta(KE)} \quad (56)$$

Let us rewrite it as

$$\boxed{KE_f = KE_i + W_{tot}} \quad (57)$$

So the total work done is converted into kinetic energy.

Let's go back to the lady dragging the suit case. From (26), (27), (28)

$$\boxed{\begin{aligned} W_{tot} &= W_g + W_n + W_{ext} + W_f = 0 + 0 + 1299 - 340 \\ &= 959\,J \end{aligned}}$$

We also know that $a_x = 3.84\ m/s^2$ Eq (24)

Suppose the suitcase started with an initial velocity v_i. Say it takes time Δt to travel the 5 m.

$$\boxed{v_f = v_i + a_x \Delta t \Rightarrow \Delta t = \frac{v_f - v_i}{a_x}}$$

$$\boxed{v_{av} = \frac{v_f + v_i}{2}}$$

Motion with constant a

$$\boxed{\Delta x = v_{av} \Delta t = \frac{v_f + v_i}{2} \times \frac{(v_f - v_i)}{a_x}}$$

$$\boxed{= \frac{v_f^2 - v_i^2}{2a_x}}$$

$$\Rightarrow \boxed{a_x \Delta x = \frac{1}{2} v_f^2 - \frac{1}{2} v_i^2}$$

multiply by m

$$\boxed{(ma_x) \Delta x = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2}$$

Since $\vec{F}_{tot} = F_{tot,x} \hat{i}$ the left hand side is the total work done.

If the suitcase started at rest

$$\boxed{\frac{1}{2} m v_f^2 = 959\,J} \Rightarrow \boxed{v_f = \sqrt{\frac{2 \times 959}{50}} = 6.2\,m/s}$$

The suitcase problem was easy because it was undergoing motion under constant acceleration.

Let's take a case when the total force, and thus the acceleration, changes.

Consider the satellite problem. Assume that its initial velocity was $\boxed{4500\ m/s = v_i}$ at $r_i = 14 \times 10^6\ m$ radially outward.

What is its final velocity when $r = r_f = 20 \times 10^6\ m$?

From 46 the work done by gravity is

$$W_g = -8.57 \times 10^8\ J$$

Since gravity is the only force on the satellite, this is the total work done.

$$\Rightarrow \boxed{KE_f = \frac{1}{2} m v_f^2 = KE_i + W_g = \frac{1}{2} m v_i^2 + W_g}$$

$$\frac{1}{2}\ 100\ kg\ v_f^2 = \frac{1}{2} \times 100\ kg \times (4.5 \times 10^3)^2 - 8.57 \times 10^8$$

$$= \frac{1}{2} \times 100 \times 20.25 \times 10^6 - 8.57 \times 10^8$$

$$= 10.125 \times 10^8 - 8.57 \times 10^8 = 1.555 \times 10^8$$

So $v_f^2 = 3.11 \times 10^6\ m^2/s^2 \Rightarrow \boxed{v_f = 1.76 \times 10^3\ m/s}$