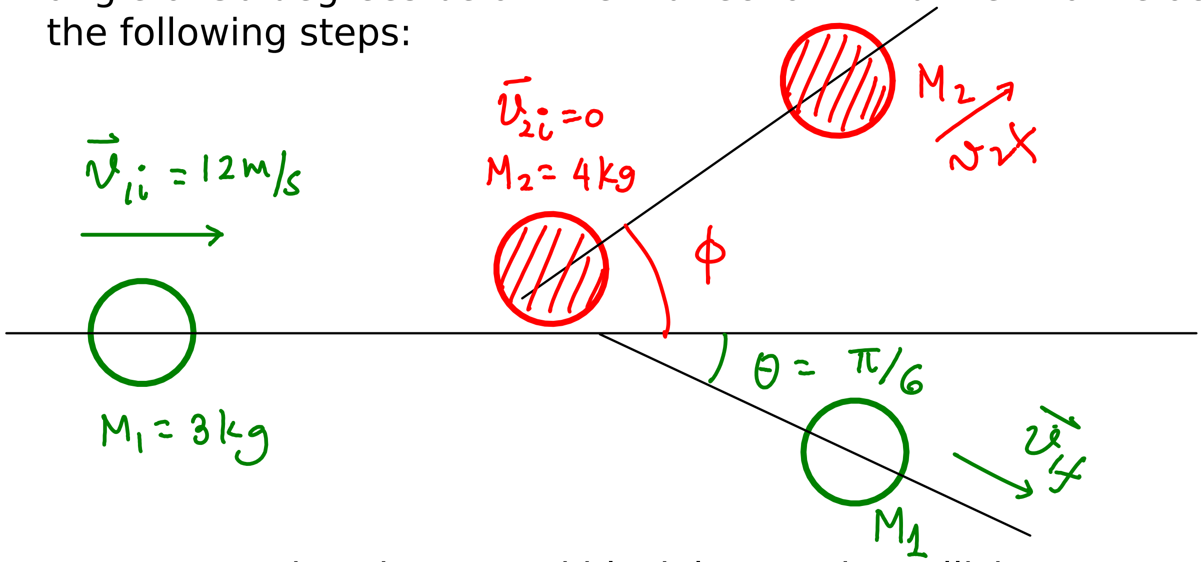
PHY231 Lecture C - Midterm 3

The last 4 pages of this test have relevant formulas. Please do not talk to anyone other than me during the test. The total points add up to 110, which means it is possible to get 110/100 on this test.

Name:

ID No:

Problem 1: A block with mass M1=3kg travelling with an initial velocity of 12m/s in the x-direction collides perfectly elastically with a stationary block of mass M2=4kg. The first block is seen to leave the collision at an angle of 30 degrees below the x-direction. Find the final velocities by using the following steps:



1a. Assume that the second block leaves the collision at an angle of phi above the x-direction. Write down the equations representing the conservation of the two components of momentum (with numbers) and the equation representing conservation of KE. **(10 points)**

$$\vec{P}_{i,tot} = \vec{P}_{1i} = M_1 \vec{v}_{1i} = M_1 v_{1i}\,\hat{i} = 36\,kg\,m/s\,\hat{i}$$

$$\vec{P}_{f,tot} = M_1 \vec{v}_{1f} + M_2 \vec{v}_{2f} = 3kg\,v_{1f}\left[\cos\tfrac{\pi}{6}\,\hat{i} - \sin\tfrac{\pi}{6}\,\hat{j}\right]$$
$$+ 4kg\; v_{2f}\left(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}\right)$$

$$\vec{P}_{i,tot} = \vec{P}_{f,tot} \Rightarrow \begin{array}{l} x \rightarrow \\ y \rightarrow \end{array} \boxed{\begin{array}{l} 36 = 3 v_{1f} \cos\tfrac{\pi}{3} + 4 v_{2f} \cos\phi \\ 0 = -3 v_{1f} \sin\tfrac{\pi}{3} + 4 v_{2f} \sin\phi \end{array}}$$

$$K_i = K_f \Rightarrow \tfrac{1}{2}\cdot 3\cdot(12)^2 = \tfrac{1}{2}\cdot 3\cdot v_{1f}^2 + \tfrac{1}{2}\cdot 4\,(v_{2f})^2$$

$$\text{or} \qquad \boxed{216 = \tfrac{3}{2} v_{1f}^2 + 2 v_{2f}^2}$$

1b. Manipulate the equations (by judiciously squaring and adding, etc) to eliminate the angle phi, and thereby find the final speeds. **(15 points)**

$$\boxed{\begin{aligned} 4v_{2f}\cos\phi &= 36 - 3v_{1f}\frac{\sqrt{3}}{2} \\ 4v_{2f}\sin\phi &= \frac{3}{2}v_{1f} \end{aligned}}$$

Square and add

$$\boxed{16v_{2f}^2 = \left(36 - 3v_{1f}\frac{\sqrt{3}}{2}\right)^2 + \frac{9}{4}v_{1f}^2}$$

From the conservation of KE $\quad 2v_{2f}^2 = 216 - \frac{3}{2}v_{1f}^2$

so $\quad 8\left(216 - \frac{3}{2}v_{1f}^2\right) = 1296 - 72\times 3\cdot v_{1f}\frac{\sqrt{3}}{2} + \frac{27}{4}v_{1f}^2 + \frac{9}{4}v_{1f}^2$

or $\quad 1728 - 12v_{1f}^2 = 1296 - 108\sqrt{3}\,v_{1f} + 9v_{1f}^2$

or $\quad 21v_{1f}^2 - 108\sqrt{3}\,v_{1f} - 432 = 0$

$$v_{1f} = \frac{108\sqrt{3} \pm \sqrt{(108)^2\times 3 + 4\times 21\times 432}}{42} = \frac{187 \pm 267}{42}$$

choose $+$.
$$\boxed{v_{1f} = 10.81\,\text{m/s}}$$

Now plug in $\quad 2v_{2f}^2 = 216 - \frac{3}{2}v_{1f}^2 = 40.72\,(\text{m/s})^2$

$$\boxed{v_{2f} = 4.51\,\text{m/s}}$$

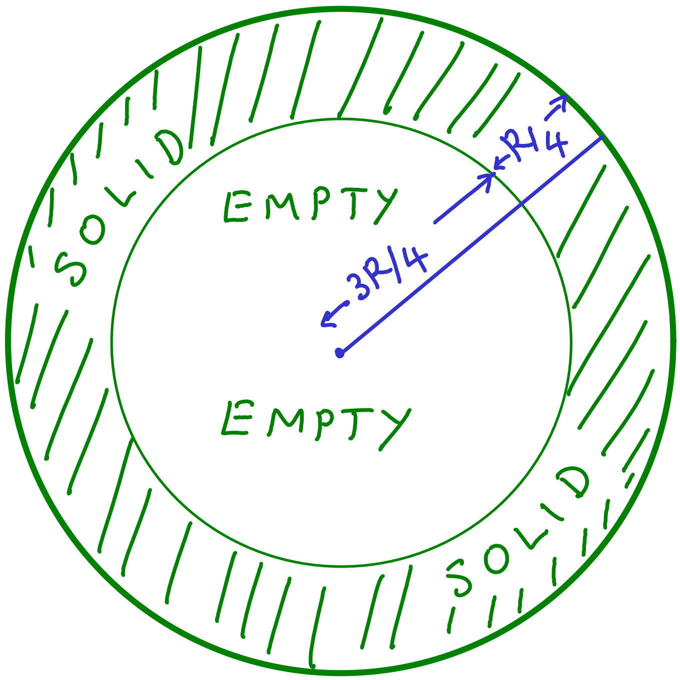
1c. Find the angle phi. **(10 points)**

$$\frac{3}{2} v_{1f} = 4 v_{2f} \sin\phi$$

$$\Rightarrow \quad \sin\phi = \frac{3}{8} \quad \frac{10.81}{4.51} = 0.9$$

$$\boxed{\phi = 1.117 \text{ rad} = 64^{\circ}}$$

Problem 2: A sphere of mass M=20kg and radius R=0.2m is partly hollow. Specifically, it has a spherical cavity at the center of radius 3R/4, so that the thickness of the spherical shell is R/4. Initially it is held against a spring of force constant k=3000N/m, compressed to 0.1m as shown. Assume that when the sphere has been released by the spring, it is not rotating. The sphere then skids on a horizontal surface until rolls without slipping. Analyze the problem using these steps.



This is the cross-section of the hollow sphere

2a. Find the moment of inertia of the hollow sphere by any method of your choice.

(10 points)

Think of the sphere as a positive mass sphere of Mass $\boxed{M_2 = \rho \cdot \frac{4}{3}\pi R_2^3}$ and a negative mass sphere of mass $\boxed{M_1 = -\rho \frac{4}{3}\pi R_1^3}$

$$\Rightarrow \boxed{\begin{aligned} I_{tot} &= \frac{2}{5} M_1 R_1^2 + \frac{2}{5} M_2 R_2^2 \\ &= \frac{2}{5} \rho \frac{4}{3}\pi (R_2^5 - R_1^5) \end{aligned}}$$

$$\boxed{\rho = \frac{M}{\frac{4}{3}\pi (R_2^3 - R_1^3)}}$$

$$I_{tot} = \frac{2}{5} M \frac{(R_2^5 - R_1^5)}{R_2^3 - R_1^3} = \frac{2}{5} \times 20\,\text{kg} \cdot \frac{[(0.2)^5 - (0.15)^5]}{(0.2)^3 - (0.15)^3}$$

$$= 8\,\text{kg}\,\text{m}^2 \times 5.277 \times 10^{-2} = \boxed{0.422\,\text{kg}\,\text{m}^2}$$

For future reference $\boxed{\frac{I}{MR^2} = \frac{0.422}{20 \times 0.04} = 0.528}$

If you had considered the shell as a "thin" spherical shell, you would have obtained

$$\frac{I}{MR^2} = \frac{2}{3} = 0.66 .$$

2b. Use the conservation of mechanical energy to find the CM velocity of the sphere when it leaves the spring. Remember, it is NOT rotating at this point. **(10 points)**

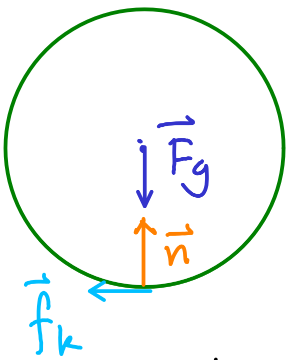
$$\boxed{E_{mech,i} = \frac{1}{2}k(x'_i)^2 = \frac{1}{2}\left(3000\frac{N}{m}\right)(0.01\,m^2) = 15\text{ Joules}}$$

$$\boxed{E_{mech,1} = \frac{1}{2}Mv_1^2 = 10\,v_1^2}$$

$$v_1^2 = 1.5\left(\frac{m}{s}\right)^2 \Rightarrow \boxed{v_1 = 1.225\,m/s}$$

2c. Friction will cause the sphere to skid until it starts rolling without slipping. Use the Impulse-momentum theorem in both linear and angular forms to find the CM velocity of the hollow sphere when it starts rolling without slipping on the horizontal plane. You will need moment of inertia of the hollow sphere to do this part. If you are not sure you have done 2a correctly, use MR^2 for this. **(20 points)**

While it skids the FBD is



$a_y = 0 \Rightarrow \quad F_g = n$

Only $\vec{f_k}$ produces torque around the CM.

Impulse in x-direction $= \boxed{\mathcal{J} = -\int_{t_1}^{t_2} f_k \, dt}$

Angular impulse $= \boxed{\mathcal{J}_{ang} = -R \int_{t_1}^{t_2} f_k dt = R\mathcal{J}}$

Let the CM speed when it rolls w/o slipping be v_2 $\Rightarrow \boxed{\omega_2 = -\frac{v_2}{R}}$

So $\boxed{\mathcal{J} = \Delta p_x = M(v_2 - v_1)}$

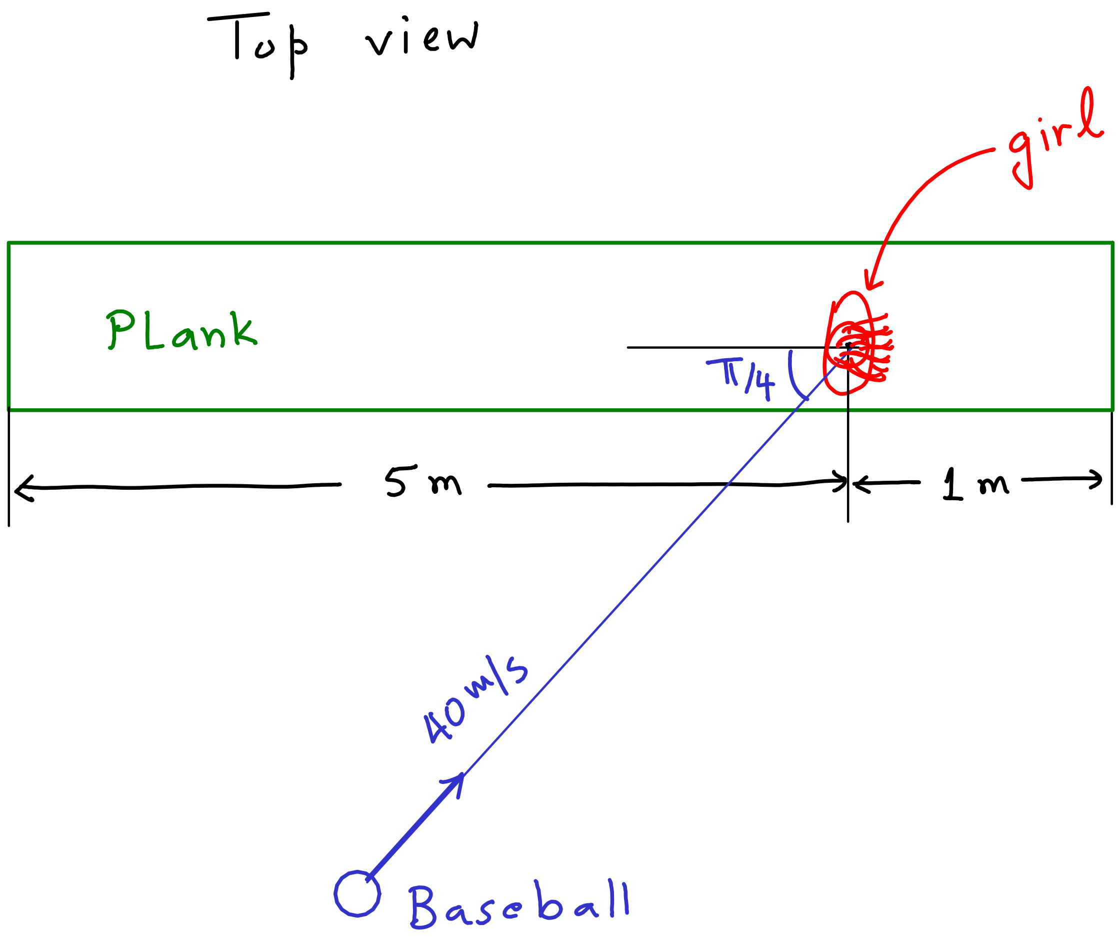
$\boxed{\mathcal{J}_{ang} = \Delta L = I\omega_2 = -I\frac{v_2}{R} = R\mathcal{J}}$

$\Rightarrow \quad \mathcal{J} = -\frac{I v_2}{R^2} = M(v_2 - v_1)$

$\left(M + \frac{I}{R^2}\right) v_2 = M v_1$ or $\boxed{v_2 = \frac{v_1}{1 + \frac{I}{MR^2}} = \frac{v_1}{1.528} =}$

$\boxed{0.802 \text{ m/s}}$

Problem 3: A girl of mass 40kg stands as shown on a plank of length 6m, and mass 40kg. The plank rests on the surface of a smooth and frozen lake. A friend on the shore throws her a baseball (mass 0.5kg) at a velocity of 40m/s in the direction shown. The girl on the plank catches the ball. You will find the subsequent motion of the plank by following these steps.

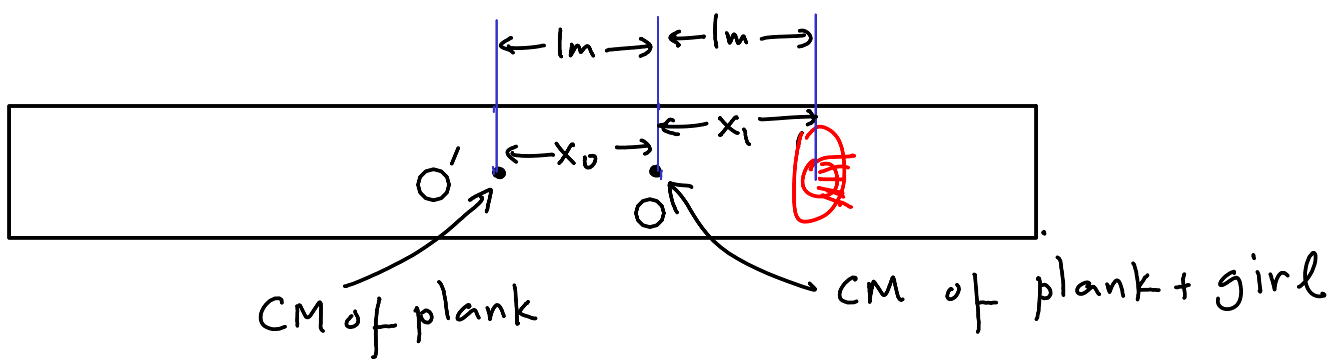


3a: Treating the plank as a thin bar, and the girl as a mass point, find the CM of the plank+girl, and its moment of inertia about an axis perpendicular to the ground, and around its CM. **(10 points)**

let the initial origin O′ be the CM of the plank. With respect to this the CM of plank + girl is given by

$$x_{CM} = \frac{M_{pl} \cdot 0 + M_{girl} \times 2m}{M_{pl} + M_{girl}} = \frac{40kg \times 2m}{80kg} = 1m$$

Now. with respect to this origin



$$I = I_{pl} + I_{girl} = \left(I_{pl,CM} + M_{pl} x_0^2\right) + M_{girl} \cdot x_1^2$$

$$= \frac{1}{12} M_{pl} \ell^2 + M_{pl} x_0^2 + M_{girl} x_1^2$$

$$= \frac{1}{12} \times 40kg \cdot 36m^2 + 80\,kgm^2 = 200\,kg\,m^2$$

3b. Find the total initial linear momentum (both components) and the total initial angular momentum. In finding the angular momentum, treat the CM of the girl + plank as the rotation axis. Remember this the CM you found in 3a. **(10 points)**

Initial momentum of baseball

$$\vec{p}_i = m_b \vec{v}_{b,i} = 20 \text{ kg m/s} \left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}} \right)$$

Initial angular momentum w.r.t. CM of plank+gire

$$\bar{L}_i = \bar{r} \times \bar{p}_i = 1m \, \hat{i} \times 20 \text{ kg} \frac{m}{s} \left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}} \right)$$

$$= \hat{k} \; 10\sqrt{2} \text{ kg m}^2/\text{s}$$

3c. Ignore the change in the position of the CM due to the addition of the ball to the system, and ignore the mass of the baseball in what follows. Use the conservation of linear and angular momentum to find the final CM velocity of the plank + girl, and its angular velocity around the CM.

(15 points)

Both momentum and angular momentum are conserved.

$$\vec{P}_f = (M_{pl} + M_{girl})\,\vec{v}_{CM,f} = 80\text{ kg}\,\vec{v}_{CM,f}$$

$$\Rightarrow \vec{v}_{CM,f} = \frac{1}{4}\frac{m}{s}\left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}\right)$$

$$L_f = I_{tot}\,\omega_f = L_i = 10\sqrt{2}\text{ kg m}^2/s$$

$$200\text{ kg m}^2\,\omega_f = 10\sqrt{2}\text{ kg m}^2/s$$

$$\omega_f = \frac{1}{10\sqrt{2}}\text{ rads}/s = 0.071\text{ rads}/\text{sec}$$

Useful formulas

Momentum is conserved in collisions. For two bodies M_1, M_2, with initial velocities $\vec{v}_{1i}$, $\vec{v}_{2i}$, and final velocities $\vec{v}_{1f}$, $\vec{v}_{2f}$

$$M_1 \vec{v}_{1i} + M_2 \vec{v}_{2i} = M_1 \vec{v}_{1f} + M_2 \vec{v}_{2f}$$

If the collision is perfectly elastic then KE is also conserved

$$\frac{1}{2} M_1 v_{1i}^2 + \frac{1}{2} M_2 v_{2i}^2 = \frac{1}{2} M_1 v_{1f}^2 + \frac{1}{2} M_2 v_{2f}^2$$

If the collision takes a time Δt then the average acceleration suffered by 1 & 2 are

$$\vec{a}_{av,1} = \frac{\vec{v}_{1f} - \vec{v}_{1i}}{\Delta t} \qquad \vec{a}_{av,2} = \frac{\vec{v}_{2f} - \vec{v}_{2i}}{\Delta t}$$

These are vectors!!

Work-Energy Theorem: $W_{tot} = \Delta K = K_f - K_i$

and $W_{nc} = \Delta E_{mech} = \Delta (K + U)$

$U_g = Mgh$ (have to choose a zero of h)

$U_s = \frac{1}{2} k x'^2$ k = Force constant of spring

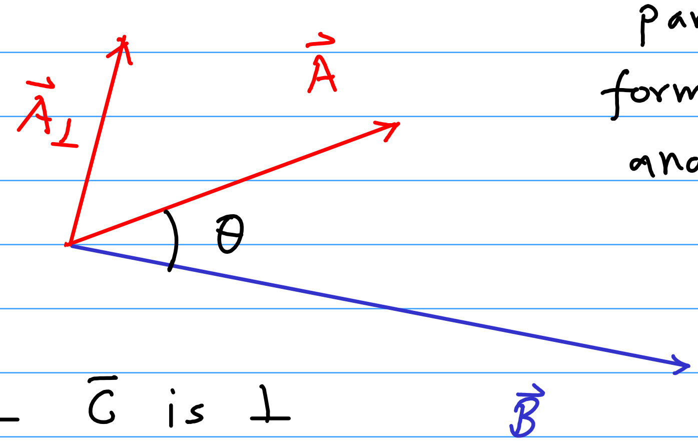
x' = Extention or compression of spring

Analogy between linear and angular motion

Position $x(t)$ (meters)	Angular position $\theta(t)$ (radians)
velocity $v(t) = \frac{dx}{dt}$	Angular velocity $\omega(t) = \frac{d\theta}{dt}$
acceleration $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	Angular acc. $\alpha(t) = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
Force F	Torque τ
Mass M	Moment of Inertia I
$F_{tot} = Ma$	$\tau_{tot} = I\alpha$
Momentum Mv	Angular momentum $L = I\omega$
Impulse $\mathcal{J} = \int_{t_i}^{t_f} F\,dt$	Angular Impulse $\mathcal{J}_{ang} = \int_{t_i}^{t_f} \tau\,dt$
Impulse - Momentum Thm	
$\mathcal{J}_{tot} = \Delta p$	$\mathcal{J}_{ang,tot} = \Delta L$

The cross-product of two vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C} = \vec{A} \times \vec{B}$

$$|\vec{C}| = A\,B\,\sin\theta = A_\perp B = B_\perp A = \text{Area of parallelogram formed by } \vec{A} \text{ and } \vec{B}$$



The direction of $\vec{C}$ is $\perp$ to the plane formed by $\vec{A}$ and $\vec{B}$ and is given by the right hand rule.

Algebraically if $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ and

$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$

$$\vec{A} \times \vec{B} = \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z) + \hat{k}(A_x B_y - A_y B_x)$$

The triple product identity says

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

This is the volume of the parallelepiped formed by $\vec{A}, \vec{B}, \vec{C}$.

Moment of Inertia of a collection of point masses M_β at locations $\bar{r}_\beta$ at distances $\delta(\bar{r}_\beta)$ from the axis of rotation $\hat{n}$ is

$$I = \sum_\beta M_\beta (\hat{n} \times \bar{r}_\beta)^2 = \sum_\beta M_\beta (\delta(\bar{r}_\beta))^2$$

Hoop around symmetry axis $\quad I_{CM} = MR^2$

Solid disk around symmetry axis $\quad I_{CM} = \frac{1}{2} MR^2$

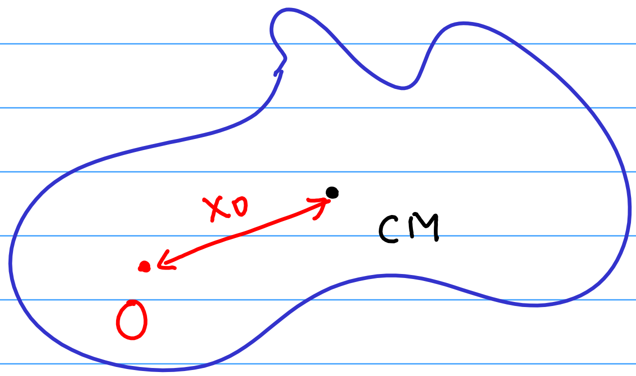
Solid sphere around symm. axis $\quad I_{CM} = \frac{2}{5} MR^2$

Thin spherical shell around symm axis $\quad I_{CM} = \frac{2}{3} MR^2$

Thin bar of length ℓ around CM and axis $\perp$ to the bar $\quad I_{CM} = \frac{1}{12} M\ell^2$

Parallel axis thm

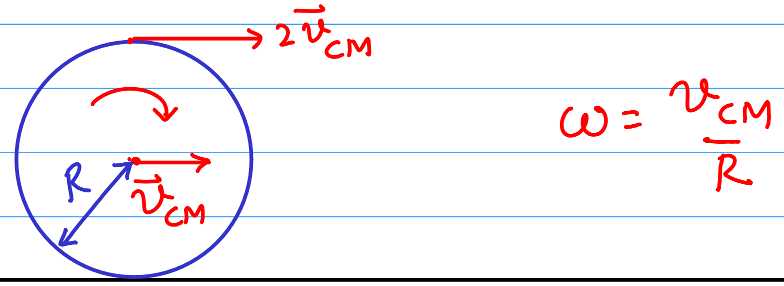
$$I_0 = I_{CM} + M x_0^2$$



Total KE of a rigid body with CM velocity $\bar{v}_{CM}$, rotating about an axis through the CM at angular velocity ω is

$$K = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I \omega^2$$

When an object of circular cross-section and radius R rolls without slipping on a plane surface with CM velocity v_{CM}



For a point mass angular momentum is

$$\vec{L} = \vec{r} \times \vec{p} = m\, \vec{r} \times \vec{v}$$

Torque due to a force $\vec{F}$ acting at pt $\vec{r}$ is

$$\vec{\tau} = \vec{r} \times \vec{F}$$

In a collision, linear momentum $\vec{p}_{tot}$ and angular momentum $\vec{L}$ are conserved

Linear impulse-momentum theorem

$$\vec{J}_{tot} = \int_{t_i}^{t_f} \vec{F}_{tot}\, dt = \Delta \vec{p}$$

Angular impulse-momentum theorem (fixed axis)

$$J_{tot,ang} = \int_{t_i}^{t_f} \tau_{tot}\, dt = \Delta L = I(\omega_f - \omega_i)$$