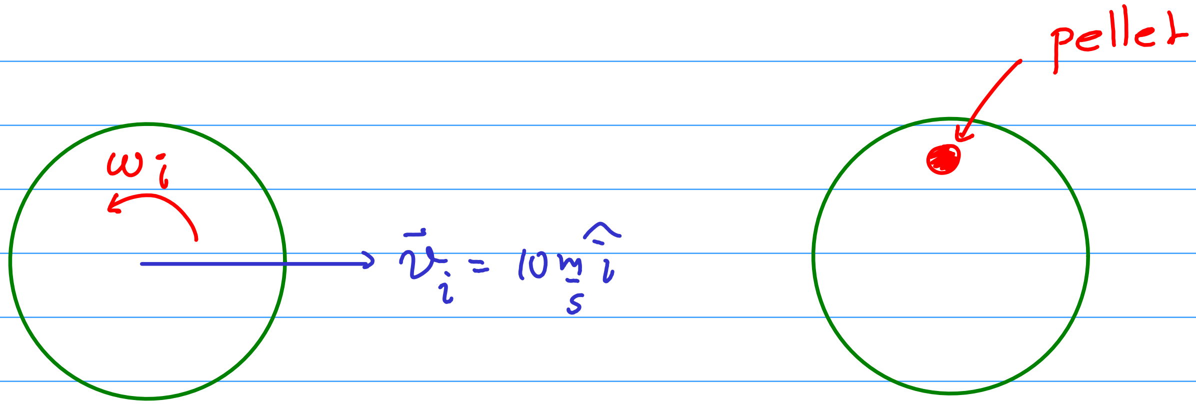
Potential Midterm Problems on Angular Collisions

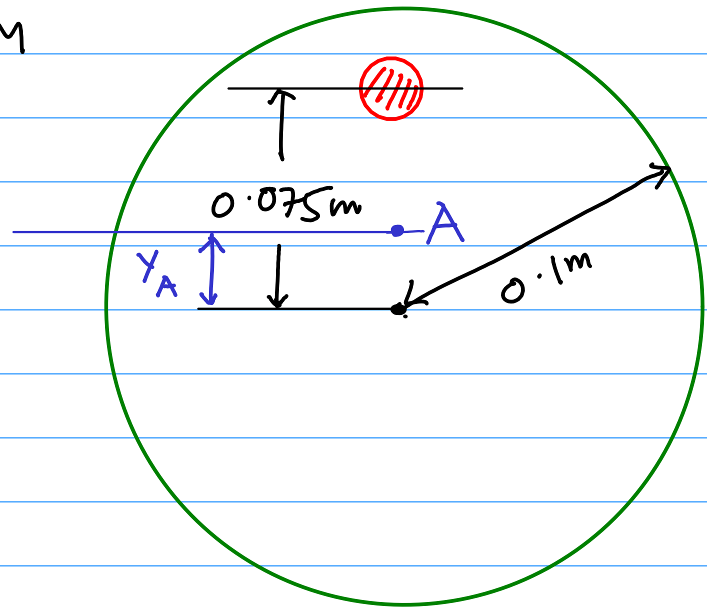
The key idea here is that in an "angular" collision where no external torques act the angular momentum is conserved.

Recall that when no external forces act on a system, the total linear momentum is conserved.

Example 1: A solid disk of mass $\boxed{M = 2\,kg}$ 1 and radius $\boxed{0.1\,m}$ 2 is sliding on a frictionless surface with a CM velocity of $\boxed{\vec{v}_i = 10\,m/s\,\hat{i}.}$ 3 Its initial angular velocity is $\boxed{\omega_i = 30\,rad/s.}$ 4 A pellet of mass $\boxed{0.5\,kg}$ 5 falls vertically on to it and sticks at a point $\boxed{0.075\,m}$ 6 from the center of the disk. Find the subsequent motion of the disk.



(1a) Find the CM of the system of disk + pellet, and the moment of inertia around the CM



Let A be the CM

By the definition of CM (7)

$$\boxed{Y_A = \frac{2kg\,(0m) + 0.5kg\,(0.075m)}{2kg + 0.5kg} = 0.015\ m}$$

Now the moment of inertia around A needs the parallel-axis theorem.

(8)

$$I_{A,tot} = I_{A,disk} + I_{A,pellet}$$

$$= \left(I_{0,disk} + M_{disk}\,Y_A^{\,2}\right) + 0.5kg\,(0.075-0.015)^2$$

$$= \frac{1}{2}(2kg)(0.1m)^2 + 2kg \times (0.015m)^2$$

$$+\ 0.5kg\,(0.06m)^2$$

$$\boxed{= 0.01225\ kgm^2}$$ (9)

(1b) Now the CM momentum and angular momentum around the CM should be conserved. Initial linear momentum is

$$\boxed{\vec{P}_{i,tot} = M_{disk}\,\vec{v}_i = 20\,kg\,\frac{m}{s}\,\hat{i}}$$ (10)

let the final CM velocity be $\vec{v}_f$ (11)

$$\boxed{\vec{P}_{f,tot} = (M_{disk} + M_{pellet})\,\vec{v}_f = 2.5\,kg\,\vec{v}_f}$$

$$\Rightarrow \boxed{\vec{v}_f = 8\,\frac{m}{s}\,\hat{i}}$$ (12)

(1c) Now we need to find the initial angular momentum. The crucial point is that this has to be computed around the new CM, which is $\boxed{y_A = 0.015\,m}$ (13) above O, because this is the pt that moves with constant velocity.

So
$$\boxed{\begin{aligned}\vec{L}_i &= \vec{L}_{disk,i} = \vec{L}_{i,disk,trans} + \vec{L}_{i,disk,rot} \\ &= \vec{R}_{CM} \times \vec{P}_{CM} + I_{disk,CM}\,\omega_i\,\hat{n}\end{aligned}}$$ (14)

$$\boxed{\hat{n} = \hat{k} \perp \text{to page}}$$ (15)

$$\vec{R}_{CM} \times \vec{P}_{CM} = \,?$$

First it is counterclockwise because O is below A $\Rightarrow$ $\boxed{\text{direction} = +\hat{n}}$ (16)

$$\boxed{\begin{aligned}|\vec{R}_{CM} \times \vec{P}_{CM}| &= y_A\,P_i = (0.015\,m)\,20\,kg\,m/s \\ &= 0.3\,kg\,m^2/s\end{aligned}}$$ (17)

$$L_{i,disk,rot} = I_{disk,CM}\,\omega_i \quad (18)$$

$$= \frac{1}{2}(2kg)(0.1m)^2\, 30\frac{rads}{s} = 0.3\ kg\,m^2/s$$

$$\vec{L}_i = \hat{k}\ 0.6\ kg\,m^2/s \quad (19)$$ adding both

Finally,

$$L_f = I_{tot}\,\omega_f \quad (20)$$

$$= 0.01225\,\omega_f$$

$$\Rightarrow \quad 0.6\ kg\,m^2/s = 0.01225\ kg\,m^2\ \omega_f \quad (21)$$

$$\omega_f = 48.98\ rad/s \quad (22)$$

Look at total KE.

$$K_i = \frac{1}{2}M_{disk}v_i^2 + \frac{1}{2}I_{disk,CM}\,\omega_i^2 \quad (23)$$

$$= \frac{1}{2}\,2kg\ 100\frac{m^2}{s^2} + \frac{1}{2}\,\frac{1}{2}(2kg)(0.01m^2)\ 900\frac{rad^2}{s^2}$$

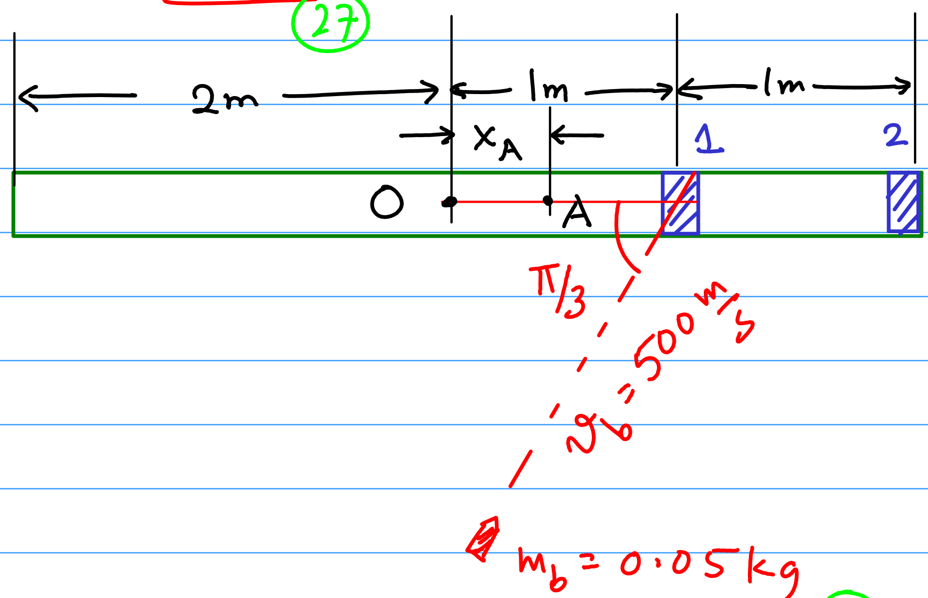
$$= 104.5\ J$$

$$K_f = \frac{1}{2}(2.5kg)(8\ m/s)^2 + \frac{1}{2}(0.01225\ kg\,m^2)\left(48.98\frac{rad}{s}\right)^2$$

$$= 94.7\ J \quad (24)$$

25

Example 2: A plank of mass $\boxed{20\,kg}$ and length $\boxed{4\,m}$ 26 has two blocks, each of mass $\boxed{5\,kg}$ 27 attached as shown.



A bullet of mass $\boxed{m_b = 0.05\,kg}$ 28 travelling at 29 $\boxed{500\,m/s}$ strikes the 1st block as shown and embeds itself. You will find the subsequent motion of the plank.

(2a) Find the CM of the plank + blocks and the moment of inertia around the CM. Treat the blocks as mass points and the plank as a thin bar.

Let A be the CM of the plank + blocks
By the definition of CM

$$\boxed{x_A = \frac{20\,kg\,(0m) + 5kg\,(1m) + 5kg\,(2m)}{20kg + 5kg + 5kg}} \quad 30$$

$$\boxed{x_A = 0.5m} \quad 31$$

To find the moment of inertia we need the parallel axis theorem

$$\boxed{\begin{aligned} I_{A,tot} &= I_{A,plank} + I_{1A} + I_{2A} \\ &= (I_{O,plank} + M_{plank}\, x_A^2) + 5\,\text{kg}\,(0.5\,\text{m})^2 \\ &\quad + 5\,\text{kg}\,(1.5\,\text{m})^2 \end{aligned}}$$ (32)

$$= \frac{1}{12}(20\,\text{kg})(4\,\text{m})^2 + 20\,\text{kg}\,(0.5\,\text{m})^2 + 1.25\,\text{kg}\,\text{m}^2 + 11.25\,\text{kg}\,\text{m}^2$$

$$= \boxed{44.17\,\text{kg}\,\text{m}^2}$$ (33)

(26) Find the final CM velocity of plank + blocks

Linear momentum is conserved in a collision

$$\boxed{\vec{P}_{i,tot} = m_b \vec{v}_b = 25\,\text{kg}\,\frac{\text{m}}{\text{s}}\left(\frac{\hat{i}}{2} + \frac{\sqrt{3}}{2}\hat{j}\right)}$$ (34)

Ignore the mass of the bullet

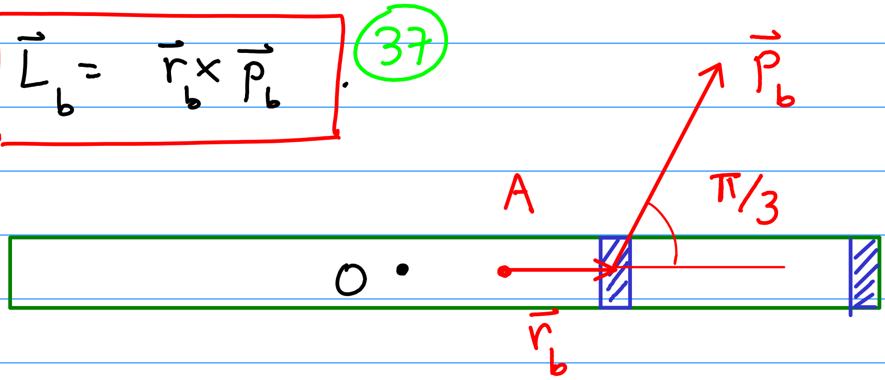
$$\boxed{\vec{P}_{f,tot} = 30\,\text{kg} \cdot \vec{v}_f}$$ (35)

$$\Rightarrow \boxed{\vec{v}_f = \frac{5}{6}\,\text{m/s}\left(\frac{\hat{i}}{2} + \frac{\sqrt{3}}{2}\hat{j}\right)}$$ (36)

(2c) Find the final angular velocity of plank + blocks

Angular momentum is conserved in a collision.
We need the initial angular momentum around the CM of the plank + blocks

$$\boxed{\vec{L}_b = \vec{r}_b \times \vec{p}_b}$$ (37)



direction is counterclockwise

magnitude

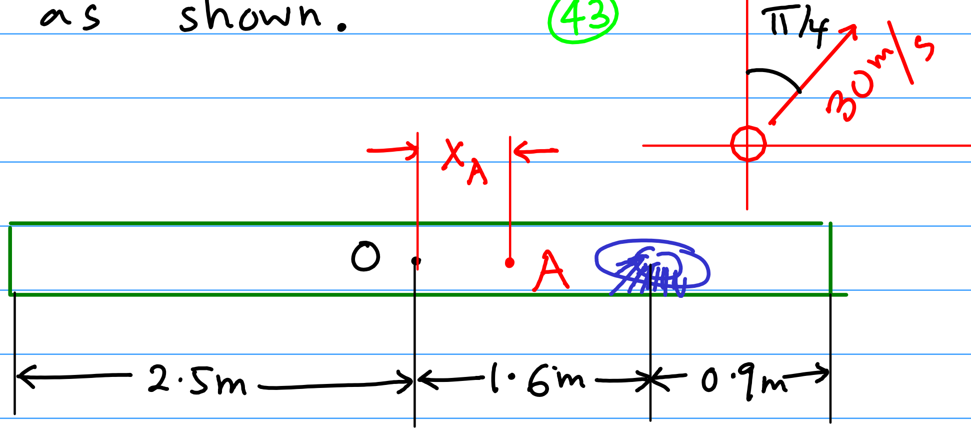
$$\boxed{L_i = r_b\, p_b \sin\frac{\pi}{3}} = 0.5\,\text{m} \cdot 25\,\text{kg}\,\frac{\text{m}}{\text{s}}\,\frac{\sqrt{3}}{2}$$ (38)

$$\boxed{L_i = 10.83\ \text{kg m}^2/\text{s}}$$ (39)

$$\boxed{L_f = I_{rot,A}\,\omega_f = 44.17\ \text{kg m}^2\ \omega_f}$$ (40)

$$\Rightarrow \boxed{\omega_f = \frac{10.83\ \text{kg m}^2/\text{s}}{44.17\ \text{kg m}^2} = 0.245\ \frac{\text{rad}}{\text{s}}}$$ (41)

Example 3: A girl (mass $\boxed{M_g = 35\text{kg}}$) stands (42) on a plank of mass $\boxed{M_p = 45\text{kg}}$ and length 5m as shown. (43)



The plank rests on the frictionless horizontal surface of a frozen pond. The girl throws a snowball of mass $\boxed{0.5\text{kg}}$ (44) with a speed $\boxed{30 \text{ m/s}}$ (45) at an angle of $\boxed{\pi/4}$ (46) w.r.t. the x-axis. Find the subsequent motion of the plank + girl.

(3a) Find the CM and the moment of inertia around the CM. Let the CM be at A

Then

$$\boxed{\begin{aligned} x_A &= \frac{45\text{kg}\,(0\text{m}) + 35\text{kg}\,(1.6\text{m})}{45\text{kg} + 35\text{kg}} \\ &= 0.7\text{m} \end{aligned}}$$ (47)

To find the moment of inertia around A we use the parallel axis theorem and treat the girl as a mass point

$$\boxed{I_{tot,A} = I_{plank,A} + I_{girl,A} = (I_{plank,CM} + M_p x_A^2) + 35\,kg\,(1.6\,m - 0.7\,m)^2}$$ 48

$$= \frac{1}{12}\,45\,kg\,(5\,m)^2 + 45\,kg\,(0.7\,m)^2 + 35\,kg\,(0.9\,m)^2$$

$$\boxed{I_{tot,A} = 144.15\ kg\,m^2}$$ 49

3b Find the final CM velocity of the girl + plank after she throws the snowball.

Initially, everything is at rest.

$$\boxed{\vec{P}_{i,tot} = 0}$$ 50

By momentum conservation

$$\boxed{\vec{P}_{f,tot} = \vec{P}_{f,P+g} + \vec{P}_{f,snowball} = 0}$$ 51

$$\boxed{\begin{aligned}\vec{P}_{f,snowball} &= (0.5\,kg)(30\,m/s)\left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}\right)\\ &= 15\,kg\,m/s\left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}\right)\end{aligned}}$$ 52

So

$$\boxed{\vec{P}_{f,P+g} = (M_p + M_g)\,\vec{v}_f = -\vec{P}_{f,snowball}}$$ 53

$$\Rightarrow \boxed{\vec{v}_f = \frac{-15\,kg\,m/s}{80\,kg}\left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}\right) = -0.1875\,\frac{m}{s}\left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}\right)}$$ 54

(3c) Find the final angular velocity of the plank + girl around their CM.

Since nothing moves initially

$\boxed{\vec{L}_{i,tot} = 0}$ (55)

Finally $\boxed{\vec{L}_{f,tot} = \vec{L}_{f,P+g} + \vec{L}_{f,snowball} = 0}$ (56)

$\boxed{\vec{L}_{f,\,snowball} = \vec{r} \times \vec{p}\,.}$ (57)

Taking the origin to be A, the CM,

$\boxed{\vec{r} = 0.9\,m\ \hat{i}}$ (58) $\boxed{\vec{p} = 15\ kg\frac{m}{s}\left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}\right)}$ (59)

$\Rightarrow$ $\boxed{\vec{L}_{f,snowball} = 13.5\,kg\,\frac{m^2}{s}\,\frac{1}{\sqrt{2}}\,\hat{k} = 9.55\,kg\,\frac{m^2}{s}\,\hat{k}}$ (60)

direction is counterclockwise

$$\boxed{\begin{aligned} L_{f,\,P+g} &= -L_{f,snowball} = -9.55\,kg\,\frac{m^2}{s} \\ &= I_{tot,A}\,\omega_f = 144.15\,kg\,m^2\,\omega_f \end{aligned}}$$ (61)

$\Rightarrow$ $\boxed{\omega_f = -0.066\ rads/sec}$ (62)

Clockwise.