Ch 7: **CIRCULAR MOTION**

So far we've done:

1) **LINEAR (STRAIGHT LINE) MOTION** *(1-D)*
2) **PROJECTILE MOTION - TRAJECTORIES** *(2-D)*

And we considered "point objects".

Now we come to finite size objects.

These can rotate about an axis or axle—**circular motion**.

E.g. — the wheel!

It's a very special type of motion.

We need to look at the quantities used to describe circular motion.
Consider a disc of radius $R$.

- All the way round is $360^\circ$.
- Distance $2\pi R$.

- Half-way is $180^\circ$.
- Distance $\pi R$.

- Quarter is $90^\circ$.
- Distance $\frac{\pi R}{2}$.
WHAT IS THE LENGTH FOR $\theta$ DEGREES?

KNOWN AS THE ARC LENGTH, $S$ (IE. HOW FAR HAS AN ANT WALKED)

IF $360^\circ$ IS $S = 2\pi R$

THEN $1^\circ$ IS $S = \frac{2\pi R}{360}$

SO $\theta^\circ$ IS $S = \frac{2\pi \theta R}{360}$ (HORRIBLE!)

REWRITE AS $S = \frac{\theta R}{\left(\frac{360^\circ}{2\pi}\right)}$

WHAT IS THE ANGLE FOR WHICH $S = R$?

ANSWER: WHEN $\theta/(360^\circ/2\pi) = 1$

OR $\theta = \frac{360^\circ}{2\pi} \approx 57^\circ$
These expressions become much more elegant if we define (and use) a new unit of angular measure: the radian.

2 radians is the angle for which $s = r$.

Definition of the radian:

1 radian $\approx 57^\circ$.

If $\theta$ is in radians:

$S = R\theta$

$\theta = \frac{S}{R}$

Dimensionless — but we still say rads.
If $\theta$ is in radians:

$$s = r \theta$$

$$\theta = \frac{s}{r}$$

Dimensionless — but we still say Rads
How many radians make a full circle?

Use $s = r\theta = \theta r$ with $s = c = 2\pi r$.

So $\theta = 2\pi$ is a full circle.

So $90^\circ = \frac{\pi}{2}$ radians \((\frac{\pi}{4})\).

180° = \pi radians.

270° = \frac{3}{2}\pi radians.
SO FAR STATIONARY OBJECTS

NOW CIRCULAR MOTION

E.G. EARTH ORBITING THE SUN

OR: A ROTATING WHEEL

LET'S CONSIDER A ROTATING DISC:

WE COULD DESCRIBE ITS MOTION BY THE NUMBER OF TURNS IT MAKES PER SECOND

BUT THERE IS A MORE USEFUL DESCRIPTION!
ROTATING DISC

WHAT IS THE SPEED ON THE EDGE OF THE DISC?

IT TRAVELS $s$ IN $t$ SECONDS.

$$v = \frac{s}{t}$$

WHAT IS THE SPEED OF A POINT FURTHER IN AT $s'$?

FOR $s'$ IN THE SAME $t$.

$$v' = \frac{s'}{t}$$

BUT $s' < s$ SO $v' \neq v$. 
So if we want to describe the motion of all points on the disc, the speed \( u \) of a point is not a universal quantity.

N.B.: For straight line motion of a finite sized body, the speed \( u \) is the same for all points of the body.

But the two points at \( R \) and \( r \) do have something in common:

- The angle that both \( \theta \) and \( r \) sweep out in a given time \( \tau \) is the same.

And:

\[
\frac{\theta}{t} \text{ is the same for all points on the disc.}
\]
WE DEFINE THE ANGULAR VELOCITY \( \omega \) BY:

\[
\omega = \frac{\Delta \theta}{\Delta t}
\]

\( \Delta \theta \) IS SWEEP OUT IN TIME \( \Delta t \)

\( \Delta \theta = \theta_f - \theta_i \)

TRUE IF ROTATING AT CONSTANT \( \omega \) (ANGULAR VELOCITY).

IF \( \omega \neq \text{CONST} \) IN TIME

\[
\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t}
\]

AVGVERAGE \( \overline{\omega} = \frac{\Delta \theta}{\Delta t} \)

UNITS OF \( \omega \): RADIANS/SEC.
What is the relationship between the arc length $s$ swept out in $t$ and $\omega$?

We have: $s = r \theta$

So $\Delta s = r \Delta \theta$

Also, the speed (m/s) of a point at $P$ is:

$\omega = \frac{\Delta s}{\Delta t} = \frac{r \Delta \theta}{\Delta t} = r \omega$

$\therefore \omega = \frac{U}{r}$

$U = r \omega$  \hspace{1cm} or  \hspace{1cm} $\omega = \frac{U}{r}$

$U$ is the tangential velocity

(ie the point at $P$ stays a fixed distance $r$ from the center)
ONE MORE THING ABOUT $\omega$: 

**How long to make one complete turn, or revolution?**

$\omega$ is radians/sec ($\frac{1}{\omega}$ is sec/rad)

One revolution is $2\pi$ radians

So $2\pi \times \frac{1}{\omega}$ is time for 1 rev

$$T = \frac{2\pi}{\omega} = \text{period}$$
Angular Acceleration

If angular velocity \( \omega \) is not constant in time we can define an:

\[ \text{Angular Acceleration} = \dot{\omega} \]

If \( \dot{\omega} \) is constant (remember \( \dot{\omega} = \text{const} \))

\[ \dot{\omega} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta \omega}{\Delta t} \]

If \( \dot{\omega} \neq \text{const} \) in time:

Average \( \ddot{\omega} = \frac{\Delta \omega}{\Delta t} \)
What is the relationship between $\alpha$ and $\omega$?

Why: $\omega = \frac{v}{r}$, so $\Delta \omega = \frac{\Delta v}{v}$

So: $\alpha = \frac{\Delta \omega}{\Delta t} = \frac{1}{r} \frac{\Delta v}{\Delta t} = \frac{\alpha_t}{r}$

So: $\alpha = \frac{\alpha_t}{r}$

$\alpha_t = r\alpha$
To summarize:

**Linear Motion**  
- $t$  
- $x$ (or $s$)  
- $v$  
- $a$ (or $a_e$)  

**Circular Motion**  
- $t$  
- $\theta$  
- $\omega$  
- $a_e = r\alpha$

?  

$\Delta$  

**Units**  

- $t \, (s)$  
- $x \, (m)$  
- $v \, (m/s)$  
- $a \, (m/s^2)$  
- $\theta \, (\text{rad})$  
- $\omega \, (\text{rad/s})$  
- $\Delta \, (\text{rad/s}^2)$

**Just multiply by $r$!**
REMEMBER THE 1-D EQUATIONS FOR STRAIGHT LINE (LINEAR) MOTION?

\[
\begin{align*}
    v &= v_0 + at \\
    x &= v_0 t + \frac{1}{2} at^2 \\
    v^2 &= v_0^2 + 2ax \\
    x &= \frac{v + v_0}{2} t
\end{align*}
\]

WE CAN ALSO USE THESE FOR CIRCULAR MOTION WITH:

\[
\begin{align*}
    x &\rightarrow s \\
    v &\rightarrow v_t \\
    a &\rightarrow a_t
\end{align*}
\]

I.E.

\[
\begin{align*}
    v &= v_0 + a_t t \\
    s &= v_0 t + \frac{1}{2} a_t t^2 \\
    v^2 &= v_0^2 + 2a_t s \\
    s &= \frac{v + v_0}{2} t
\end{align*}
\]

NOT VERY USEFUL IN THIS FORM, SO WE CONVERT THEM
To convert them to contain angular quantities we use

\[
\begin{align*}
S &= r \theta \\
\mathbf{v} &= r \mathbf{w} \\
\mathbf{a}_t &= r \mathbf{a}_t
\end{align*}
\]

\( t = t! \)

So

\[ \mathbf{v} = \mathbf{v}_0 + \mathbf{a}_t t \]

becomes

\[ \mathbf{w} = \mathbf{w}_0 + \mathbf{a}_t t \]

or:

\[ \mathbf{w} = \mathbf{w}_0 + \mathbf{a}_t t \]

Next:

\[ S = \mathbf{v}_0 t + \frac{1}{2} \mathbf{a}_t t^2 \]

\[ \Rightarrow \quad \Delta \theta = \Delta \omega_0 t + \frac{1}{2} \mathbf{a}_t t^2 \]

or:

\[ \Delta \theta = \omega_0 t + \frac{1}{2} \mathbf{a}_t t^2 \]
\[ S = \frac{v_0 + v}{2} \cdot t \]

\[ \Rightarrow \theta = \frac{\omega + \omega_0}{2} \cdot t \]

\[ \theta = \omega + \omega_0 \cdot t \]

So far we've had every term contains \( \omega \) which has cancelled out.

What about

\[ v^2 = v_0^2 + 2a_\ell \cdot S \]

Which has squared terms?

Let's see. It becomes

\[ a^2 \omega^2 = a^2 \omega_0^2 + 2(\chi_a)(\chi_0 \theta) \]

Or \[ \omega^2 = \omega_0^2 + 2\chi_0 \theta \]
So for constant $a$, we have:

$$
\omega = \omega_0 + \alpha t \\
\theta = \omega_0 t + \frac{1}{2} \alpha t^2 \\
\theta = \frac{\omega + \omega_0}{2} t \\
\omega^2 = \omega_0^2 + 2 \alpha \theta
$$

And we don't have to memorize them if we notice that they have the same form as the linear equations with:

$$
x \rightarrow \theta \\
v \rightarrow \omega \\
\alpha \rightarrow \alpha
$$