Motion in a Plane

PROJECTILE MOTION

So far 1-D motion:

\[ x = v_x t, \quad a_x = a \]

Cars accelerating, etc.

Gravity

Now we will combine these: Projectile motion

\[ y = v_y t, \quad a_y = -g \]

In fact we use

\[ x, y, v_x, v_y, a_x, a_y, t \]
2) **CONSIDER THIS:**

YOU ARE TRAVELING IN A TRAIN AT 200 mph.

1) **YOU HOLD OUT A BOOK IN FRONT OF YOU AND LET IT GO.**

IT LANDS AT YOUR FEET.

2) **THE TRAIN STOPS. YOU DO THE SAME THING.**

IT LANDS AT YOUR FEET.
THE HORIZONTAL MOTION DOES NOT SEEM TO AFFECT THE VERTICAL MOTION!
Let's do two demos:

1) Fire! \( \Rightarrow \text{Spring} \Rightarrow \text{Drop} \)

Both land at the same instant.

2) \( U = \text{const} \)

Ball lands back in car.
The simplest explanation that fits the facts.

We can separate out vertical and horizontal motion.

Vertical motion given by gravity:

\[ a_y = -9 \]

Horizontal motion given by:

\[ a_x = 0 \]
For an object in the air:

If thrown from the origin:

\[ \begin{align*}
\mathbf{U}_x &= \mathbf{v}_x \\
\mathbf{U}_y &= \mathbf{v}_y \\
\mathbf{a}_x &= 0 \\
\mathbf{a}_y &= -g
\end{align*} \]
EQUATIONS OF MOTION

1) HORIZONTAL:

\[ u_x = v_{0x} + a_xt \]
\[ u_x^2 = v_{0x}^2 + 2a_x \Delta x \]
\[ x = v_{0x} t + \frac{1}{2} a_x t^2 \]
\[ x = \frac{v_{0x} + u_x}{2} \cdot t \]

2) VERTICAL:

\[ u_y = v_{0y} - gt \]
\[ u_y^2 = v_{0y}^2 - 2gy \]
\[ y = v_{0y} t - \frac{1}{2} gt^2 \]
\[ y = \frac{v_{0y} + u_y}{2} \cdot t \]
PROJECTILE MOTION - GROUND TO GROUND

STARTS FROM ORIGIN AT $t = 0$ WITH $\vec{U}_0$ AT $\theta_0$

WE WANT $\vec{r}$ AND $\vec{V}$ AT ANY LATER TIME $t$

I.E. WE WANT $x$, $y$, $U_x$, $U_y$, $V_x$, $V_y$ AT TIME $t$
PROJECTILE MOTION - GROUND TO GROUND

**INITIAL CONDITIONS:**

\[
\begin{align*}
\vec{u}_0x &= u_0 \cos \theta_0 \\
u_0y &= u_0 \sin \theta_0
\end{align*}
\]

\[t = 0\]

**To get \( \vec{u} \) at \( t \):** WE USE EQUATIONS OF MOTION:

\[
\begin{align*}
\vec{u}_x &= \vec{u}_0x \quad (= \text{const}) \\
u_y &= u_0y + ayt \\
&= u_0 \sin \theta_0 \quad -gt
\end{align*}
\]
**PROJECTILE MOTION - GROUND TO GROUND**

\[ \mathbf{U} \]

\[ \begin{align*}
U_x &= U_{0x} = v_0 \cos \theta_0 \\
U_y &= v_0 \sin \theta_0 - gt
\end{align*} \]

**THEN**

\[ \mathbf{U} = \sqrt{U_x^2 + U_y^2} = v_{CH} \]

\[ \theta = \tan^{-1}\left(\frac{U_y}{U_x}\right) \text{ or } \arctan\left(\frac{U_y}{U_x}\right) \]
**Projectile Motion - Ground to Ground**

\[ y \uparrow \]

\[ \begin{aligned} \vec{v}_y &= 0 \\ \Rightarrow \quad \vec{u} &= \vec{u}_0 \rangle \\ \text{APEX} \\ \vec{u}_y &= \Theta \\ \end{aligned} \]

**At Highest Point (Apex):**

\[ \vec{v}_y = 0 \]

**After Apex** \( \vec{v}_y \) **is negative**

**Notice That**

\[ \vec{u}_x = \vec{u}_0 \rangle \text{ for all } t \]

(Until it hits the ground!)
LETS FIND POSITION AT $t$:

**HORIZONTAL**: $(x = 0 \text{ at } t = 0)$

$$x = u_0x t = u_0 \cos \theta_0 t$$

**VERTICAL**: $(y = 0 \text{ at } t = 0)$

$$y = u_0y t - \frac{1}{2} g t^2$$

$$y = (u_0 \sin \theta_0) t - \frac{1}{2} g t^2$$
**PROJECTILE MOTION - GROUND TO GROUND**

**RANGE:** Horizontal distance traveled before hitting ground given by $x$ when $y = 0$.

$y = 0 = u_0 \sin \theta_0 t - \frac{1}{2} gt^2$

**IE:** $\frac{1}{2} gt^2 = u_0 \sin \theta_0 t$

So $y = 0$ at $t = 0$ (we knew that!)

Or $\frac{1}{2} gt = u_0 \sin \theta_0$

$t = \frac{2 u_0 \sin \theta_0}{g}$ for $y = 0$
PROJECTILE MOTION - GROUND TO GROUND

\[ y \uparrow \]

\[ \theta_0 \]

\[ \frac{v_0}{\theta_0} \]

\[ \text{RANGE} \]

\[ \text{RANGE GIVEN BY:} \]

\[ x = v_0 \cos \theta_0 t \]

\[ \text{AT } t = \frac{2v_0 \sin \theta_0}{g} \]

\[ x = \frac{v_0 \cos \theta_0 \times 2v_0 \sin \theta_0}{g} \]

\[ x = 2 \sin \theta_0 \cos \theta_0 \frac{v_0^2}{g} \]
PROJECTILE MOTION - GROUND TO GROUND

\[ \text{RANGE: } \frac{2 \sin \theta_0 \cos \theta_0 \cdot u_0^2}{g} \]

QUESTION: IF I CAN THROW A BALL WITH SOME MAXIMUM VALUE OF \( u_0 \), IS THERE AN ANGLE \( \theta_0 \) FOR WHICH THE RANGE IS THE GREATEST?

YES! WE CAN FIND IT!
PRO젝TILE MOTION - GROUND TO GROUND

Before we find the best $\theta_0$, how do I know it exists?

Consider $\theta_0 = 90^\circ$ straight up

Range = 0!

Consider $\theta_0 = 0$: hits ground instantly (it never leaves the ground)

$\rightarrow$ Range = 0!

For $\theta_0$ in between Range $\neq 0$.

$\therefore$ There must be a maximum!
**PROJECTILE MOTION - GROUND TO GROUND**

![Diagram of projectile motion with initial velocity and angle labeled]

We want \( \theta_0 \) which maximizes

\[ \frac{2 \sin \theta_0 \cos \theta_0 \, v_0^2}{g} \]

### TRIG!

\[ 2 \sin \theta_0 \cos \theta_0 = \sin 2 \theta_0 \]

### MORE TRIG.

Max value of \( \sin \phi \) is 1 when \( \phi = 90^\circ \)

\[ \therefore \text{Max range when } 2\theta_0 = 90^\circ \]

Or

Max range when \( \theta_0 = 45^\circ \)
PROJECTILE MOTION - GROUND TO GROUND

MAX. RANGE WHEN $\theta_0 = 45^\circ$
(FOR FIXED $U_0$.)

VALUE OF MAX. RANGE

$R = 2 \sin \theta_0 \cos \theta_0 \frac{U_0^2}{g}$

$= 2 \sin 45^\circ \cos 45^\circ \frac{U_0^2}{g}$

$= \frac{1}{9}$

WE IGNORED AIR RESISTANCE (TOO DIFFICULT)