7.4 Centripetal Acceleration

Figure 7.6a shows a car moving in a circular path with constant linear speed \( v \). Even though the car moves at a constant speed, it still has an acceleration. To understand this, consider the defining equation for average acceleration:

\[
\bar{a}_v = \frac{\bar{v}_f - \bar{v}_i}{t_f - t_i}
\]

[7.12]

The numerator represents the difference between the velocity vectors \( \bar{v}_f \) and \( \bar{v}_i \). These vectors may have the same magnitude, corresponding to the same speed, but if they have different directions, their difference can't equal zero. The direction of the car's velocity as it moves in the circular path is continually changing, as shown in Figure 7.6b. For circular motion at constant speed, the acceleration vector always points toward the center of the circle. Such an acceleration is called a centripetal (center-seeking) acceleration. Its magnitude is given by

\[
a_c = \frac{v^2}{r}
\]

[7.13]

To derive Equation 7.13, consider Figure 7.7a. An object is first at point \( \bullet \) with velocity \( \bar{v}_i \) at time \( t_i \) and then at point \( \circ \) with velocity \( \bar{v}_f \) at a later time \( t_f \). We assume \( \bar{v}_i \) and \( \bar{v}_f \) differ only in direction; their magnitudes are the same \( (v_i = v_f = v) \). To calculate the acceleration, we begin with Equation 7.12,

\[
\bar{a}_v = \frac{\bar{v}_f - \bar{v}_i}{t_f - t_i} = \frac{\Delta \bar{v}}{\Delta t}
\]

[7.14]

where \( \Delta \bar{v} = \bar{v}_f - \bar{v}_i \) is the change in velocity. When \( \Delta t \) is very small, \( \Delta s \) and \( \Delta \theta \) are also very small. In Figure 7.7b \( \bar{v}_i \) is almost parallel to \( \bar{v}_f \), and the vector \( \Delta \bar{v} \) is approximately perpendicular to them, pointing toward the center of the circle. In the limiting case when \( \Delta t \) becomes vanishingly small, \( \Delta \bar{v} \) points exactly toward the center of the circle, and the average acceleration \( \bar{a}_v \) becomes the instantaneous acceleration \( \bar{a} \). From Equation 7.14, \( \bar{a} \) and \( \Delta \bar{v} \) point in the same direction (in this limit), so the instantaneous acceleration points to the center of the circle.

The triangle in Figure 7.7a, which has sides \( \Delta s \) and \( r \), is similar to the one formed by the vectors in Figure 7.7b, so the ratios of their sides are equal:

\[
\frac{\Delta v}{v} = \frac{\Delta s}{r}
\]

[7.15]

or

\[
\Delta v = \frac{v}{r} \Delta s
\]

Substituting the result of Equation 7.15 into \( a_v = \Delta v/\Delta t \) gives

\[
a_v = \frac{v}{r} \frac{\Delta s}{\Delta t}
\]

[7.16]

But \( \Delta s \) is the distance traveled along the arc of the circle in time \( \Delta t \), and in the limiting case when \( \Delta t \) becomes very small, \( \Delta s/\Delta t \) approaches the instantaneous value of the tangential speed, \( \omega \). At the same time, the average acceleration \( a_v \) approaches \( a_c \), the instantaneous centripetal acceleration, so Equation 7.16 reduces to Equation 7.13:

\[
a_c = \frac{v^2}{r}
\]

Because the tangential speed is related to the angular speed through the relation \( v_t = \omega r \) (Eq. 7.10), an alternate form of Equation 7.13 is

\[
a_c = \frac{r^2 \omega^2}{r} = r \omega^2
\]

[7.17]