Comparing eqs (4)+(5):

$$\int \dots \int d\Omega_{n-1} = n C_n \qquad (6)$$

use trick to avoid direct computation:

Define function

$$f(x_1 \dots x_n) = e^{-(x_1^2 + \dots + x_n^2)} = e^{-r^2}$$

Integrating f over n-dim. space:

$$\underbrace{\int_{-\infty}^{\infty} dx_1 \dots dx_n \, e^{-(x_1^2 + \dots x_n^2)}}_{\text{over rectangular}} = \int_0^{\infty} r^{n-1} dr \int d\Omega_{n-1} \underbrace{e^{-r^2}}_{\substack{\text{independent} \\ \text{of } \Omega_{n-1}}}$$

over rectangular and azimuthal coord.

independent of Ω_{n-1} (use eq.6)

Performing $\int$ over angular part:

$$\int_{-\infty}^{\infty} e^{-x_1^2} dx_1 \dots \int_{-\infty}^{\infty} e^{-x_n^2} dx_n = n C_n \int_0^{\infty} r^{n-1} e^{-r^2} dr$$

All integrals are evaluated directly:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \; ; \quad \int_0^{\infty} r^{n-1} e^{-r^2} dr = \frac{1}{2} \Gamma(n/2)$$

$$\pi^{n/2} = C_n \frac{n}{2} \Gamma(n/2) = C_n \Gamma\left(1 + \frac{n}{2}\right)$$

used $x\Gamma(x) = \Gamma(x+1)$

$$\Rightarrow \boxed{C_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}} \qquad (7)$$

but C_n is independent of f!

Check: nC_n for $n=2,3$:

$$2C_2 = \frac{2\pi}{\Gamma(2)} = 2\pi = \int_0^{2\pi} d\theta$$

$$3C_3 = \frac{3\pi}{\Gamma(5/2)} = 4\pi = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta$$

$\longrightarrow$ Here $\Gamma(5/2) = (3/2)\,\Gamma(3/2) = \frac{3}{4}\sqrt{\pi}$

V_n and S_{n-1} of n-dim. sphere: insert (7) $\to$ (1), (3)

$$\boxed{V_n(R) = \frac{\pi^{n/2} R^n}{\Gamma(1+n/2)}} \; ; \quad S_{n-1}(R) = \frac{n\pi^{n/2} R^{n-1}}{\Gamma(1+n/2)}$$

$$\Downarrow$$

Using $\Gamma(1+n/2) = \frac{n}{2}\Gamma(n/2) \implies \boxed{S_{n-1}(R) = \frac{2\pi^{n/2} R^{n-1}}{\Gamma(n/2)}}$

$\uparrow$ (*)

Check: $n = 2, 3$: $\begin{cases} V_3(R) = \frac{4}{3}\pi R^3; \\ S_2(R) = 4\pi R^2 \end{cases}$

32)

N nonrelativistic identical particles

$$H = \frac{1}{2m}\sum_{j=1}^{Nd} P_i^2$$

d - dimensions
N - nu. of particles

Nu. of available states in the ensemble:

$$\Omega(E) = \int d^{Nd}q \int d^{Nd}P / N! h^{Nd} = \frac{V}{N!\, h^{Nd}} \int d^{Nd}P$$

where $V = V_d^N$

V_d - volume of the system

$\Rightarrow$ Now treat $\int d^{Nd}P$ in integral:

define a radius of Nd - dim. sphere:

$$R^2 \equiv K^2 \equiv 2mE = \sum_{j=1}^{Nd} P_j^2$$

schematic

$$\Rightarrow \Omega(P) : \underbrace{\int^{Nd-1}}_{\text{surface}} \otimes R^{Nd}$$

$\Rightarrow$ volume of a shell = = surface $\times \delta R$

For shell $(E, \delta E) \equiv (K, \delta K)$

$$K\delta K = m\delta E \Rightarrow \delta K = \frac{m\delta E}{K}$$

From p. 31 (eq. *): $S_{n-1} \equiv S_{Nd-1} = \frac{2\pi^{Nd/2}}{\Gamma(Nd/2)} R^{Nd-1}$

$$\Rightarrow \int d^{Nd}P = S_{Nd-1}\,\delta R = \frac{2\pi^{Nd/2} K^{Nd-1}}{\Gamma(Nd/2)}\,\delta K$$

$$\int_{\text{shell}} d^{Nd}P = \frac{2\pi^{Nd/2} K^{Nd-1}}{\Gamma(Nd/2)}\left(\frac{m\,\delta E}{K}\right)$$

Now express in terms of E:

$$\frac{K^{Nd-1}}{K} = K^{Nd-2} = (2mE)^{\frac{Nd}{2}-1} = \frac{1}{2mE}(2mE)^{Nd/2}$$

Thus,

$$\Omega(E) = \frac{V_d^N}{N!\,h^{Nd}} \frac{(2mE)^{Nd/2}}{\Gamma(Nd/2)} \frac{\delta E}{E}\, 2\pi^{Nd/2}$$

$$\Rightarrow \boxed{\Omega(E) = \frac{V_d^N}{N!\,h^{Nd}} \frac{(2mE\pi)^{Nd/2}}{\Gamma(Nd/2)} \frac{\delta E}{E}} \qquad (*)$$

Useful to write this in terms of thermal "de Broglie":

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

So

$$\log \Omega(E) = N \log V_d - \log N! + \frac{Nd}{2}\log\left(\frac{2\pi m E}{h^2}\right) - \log\Gamma(Nd/2) + \log\left(\frac{\delta E}{E}\right)$$

($\log \to \ln$)

Stirling approximation: $x \gg 1$

$$\ln\Gamma(x) = (x-1)\ln(x-1) - (x-1)$$

$$N! = \Gamma(N+1) \to \ln N! \approx N\ln N - N$$

34)

$$\ln \Omega(E) = N \ln V_d - N \ln N + N + \frac{Nd}{2} \ln\left(\frac{2\pi m E}{h^2}\right) - \left(\frac{Nd}{2} - 1\right) \ln\left(\frac{Nd}{2} - 1\right) + \frac{Nd}{2} - 1 + \ln\left(\frac{\delta E}{E}\right) =$$

$$= N \ln(V_d / N) + N\left(\frac{d}{2} + 1\right) - 1 + \frac{Nd}{2}\left[\ln\left(\frac{2\pi m E}{h^2}\right) - \ln\left(\frac{Nd}{2} - 1\right)\right] + \ln\left(\frac{Nd}{2} - 1\right) + \log\left(\frac{\delta E}{E}\right) =$$

$$= \frac{Nd}{2} \ln\left(\frac{V_d}{N}\right)^{2/d} + N\left(\frac{d}{2} + 1\right) + \frac{Nd}{2} \ln\left(\frac{2\pi m E}{h^2 N d/2}\right) + \ln\left[\frac{\delta E}{E}\left(\frac{Nd}{2} - 1\right)\right] =$$

$$= N\left\{\left(\frac{d}{2} + 1\right) + \frac{d}{2} \ln\left(\frac{2\pi m E}{h^2 N d/2}\right) + \frac{d}{2} \ln\left(\frac{V_d}{N}\right)^{2/d}\right\} + \ln\left[\frac{\delta E}{E}\left(\frac{Nd}{2} - 1\right)\right]$$

For $\left.\begin{matrix} N \gg 1 \\ V_d \to \infty \\ E \to \infty \end{matrix}\right\}$; $\frac{N}{V_d}$; $\frac{E}{N} \sim \mathcal{O}(1)$ and

in most physics systems: $\frac{\delta E}{E} = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$

If $\left|\ln\left(\frac{\delta E}{E}\right)\right| \ll N$ + Stirling formula

$\Rightarrow \delta E$ disappears!

$$\Rightarrow \boxed{\ln \Omega = N\left\{\left(\frac{d}{2} + 1\right) + \frac{d}{2} \ln \frac{2\pi m E}{\frac{Nd}{2} h^2 \left(\frac{N}{V_d}\right)^{2/d}}\right.} \to \delta E \text{ disappears!!}$$

35)

Insert: Extensive & Intensive values

Extensive values: $\sim N$

Intensive values: $\not\sim N$

For normal systems, in the thermo. limit, $N \to \infty$, with $N/V \sim$ const,

$\Rightarrow$ E and S are extensive, while T, P, μ are intensive.

But not all systems are normal!

For example: (1) self-grav. systems

(2) a system with net e-charge.