Near equilibrium

$$S_2(E_0-E_1, N_0-N_1) = S_2(E_0, N_0) - E_1 \overbrace{\left(\frac{\partial S_2}{\partial E_2}\right)_{\bar{E}_1}}^{1/T} - N_1 \overbrace{\left(\frac{\partial S_2}{\partial N_2}\right)_{\bar{N}_1}}^{-\mu/T} + \dots$$

$$= S_2(E_0, N_0) - \beta(E_1 - \mu N_1)$$

Thus,

$$\Omega(E_0, N_0) = \sum_{N_1} \int \frac{dE}{\delta E} \int \frac{dE_1}{\delta E} \Omega_1(E_1, N_1)\, e^{-\beta(E_1 - \mu N_1)}$$

$$e^{S_2(E_0, N_0)}$$

Thus probability

$$P(E_1, N_1) \propto \Omega_1(E_1, N_1)\, e^{-\beta(E_1 - \mu N_1)}$$

In analogy with canonical ensemble:

$$\boxed{Z_G = \sum_{N=0}^{\infty} \int_0^{\infty} \frac{dE}{\delta E} \Omega(E, N)\, e^{-\beta(E - \mu N)}}$$

Define grand potential:

$$\Omega_G = -\frac{1}{\beta} \ln Z_G$$

same manipulations:

$$\langle E \rangle = -\left(\frac{\partial \ln Z_G}{\partial \beta}\right)_\mu = \frac{\partial}{\partial \beta}(\beta \Omega_G)\Big|_\mu$$

$$\langle N \rangle = \frac{1}{\beta}\left(\frac{\partial \ln Z_G}{\partial \mu}\right)_\beta$$

53)

Since $\Omega e^{-\beta(E-\mu N)}$ has a maximum at some $E=\bar{E}$, $N=\bar{N}$

$\Rightarrow$ if distribution is sharp, may approximate

$$\bar{E} = \langle E \rangle \, ; \quad \bar{N} = \langle N \rangle$$

Then

$$Z_G \propto e^{-S(\bar{E},\bar{N})} e^{-\beta(\bar{E}-\mu\bar{N})}$$

$$\Rightarrow \boxed{\Omega_G = \bar{E} - T\bar{S} - \mu N} \leftarrow \text{Grand potential in thermo.}$$

Now calculate fluctuations

$$\langle (\Delta E)^2 \rangle = \left(\frac{\partial^2 \ln Z_G}{\partial \beta^2} \right)_\mu$$

Similarly, $(\Delta N)^2 = \langle N^2 \rangle - \langle N \rangle^2$

$$\langle N^2 \rangle = \frac{1}{\beta^2} \frac{1}{Z_G} \frac{\partial^2 Z_G}{\partial \mu^2}$$

Thus,

$$\langle N^2 \rangle - \langle N \rangle^2 = \frac{1}{\beta^2} \left[\frac{1}{Z_G} \frac{\partial^2 Z_G}{\partial \mu^2} - \left(\frac{1}{Z_G} \frac{\partial Z_G}{\partial \mu} \right)^2 \right]$$

$$= \frac{1}{\beta^2} \frac{\partial^2 \ln Z_G}{\partial \mu^2} = \frac{1}{\beta} \frac{\partial}{\partial \mu} \left[\frac{1}{\beta} \frac{\partial}{\partial \mu} \ln Z_G \right]$$

$$= \frac{1}{\beta} \frac{\partial \langle N \rangle}{\partial \mu}$$

$$\Rightarrow \langle (\Delta N)^2 \rangle \propto \langle N \rangle$$

and $\dfrac{\langle (\Delta N)^2 \rangle}{\langle N \rangle^2} \propto \dfrac{1}{\langle N \rangle}$ is small when $\langle N \rangle \to \infty$

Equivalence of Ensembles

Consider:

* microcanonical ens.: fixed E, V, N
* canonical ens.: fixed V, N, but E fluctuates, prob. $\propto \underbrace{e^{-\beta H}}_{\text{Gibbs factor}}$
* grandcanonical ens.: fixed V, E as in canonical, N fluctuates with weight $e^{\beta\mu}$

Equivalence of ensembles $\Big\} \Rightarrow$ Pathes to $\underbrace{\text{thermo}}_{\text{micro}\to\text{macro}}$ are equivalent

i.e., thermo limit: fluctuations in E and N $\to 0$! $\Rightarrow$ unimportant

$\Rightarrow$ **In thermo limit: all statistical prob. distributions behave the same way**

Density of states $\rho(E)$: $$\frac{\Omega(E)}{\delta E} = \sum_r \delta(E_r - E) \quad \begin{cases} \delta = 1 \text{ when } E < E_r < E + \delta E \\ \delta = 0 \text{ when } E_r \text{ is outside the range} \end{cases}$$

Rewrite: $$\frac{\Omega(E)}{\delta E} = \sum_r \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(E-E_r)\alpha}\, d\alpha}_{\delta(E-E_r)}$$

Can multiply by $\int_{-\infty}^{\infty} d\alpha\, e^{i(E-E_r)\alpha}\, \underbrace{f(E-E_r)}_{f=1 \text{ when } E-E_r=0}$

55) Consider $\omega = i\alpha + \beta \Rightarrow d\alpha = \frac{d\omega}{i}$

Changing variables: $\alpha \to \omega$

$$\frac{\Omega(E)}{\delta E} = \frac{1}{2\pi i} \sum_r \int_{-i\infty+\beta}^{i\infty+\beta} d\omega \, e^{(E-E_r)\omega} \, \underbrace{e^{-\beta(E-E_r)}}_{=1, \text{ when } E-E_r=0}$$

However, $\sum_r e^{-\omega E_r} = Z(\omega) \to$ canonical ensemble $(*)$

$$\boxed{\Omega(E) = \frac{\delta E}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} d\omega \, e^{\omega E} Z(\omega)}$$

$\hookrightarrow$ inverse Laplace transform

if $\alpha = 0 \Rightarrow$ all terms in $(*)$ are >0

if $\alpha \neq 0 \Rightarrow$ oscillatory terms $e^{-i\alpha E_r}$ add terms $\Rightarrow$ with $\pm$ and

$\hookrightarrow$ so only region close to $\alpha = 0$ contributes to the integral!

Laplace Transform. Theory

Two examples for $\underbrace{\int_0^\infty f(t)e^{-zt}dt}_{\text{Laplace integral}} = \underbrace{\lim_{T\to\infty}}_{\text{if this exists}} \int_0^T f(t)\, e^{-zt} dt$

Example 1: $f(t)=1$

$$\mathcal{L}(1) = \int_0^\infty e^{-zt}dt = \lim_{T\to\infty}\int_0^T e^{-zt}dt = \lim_{T\to\infty}\left[-\frac{e^{-zt}}{z}\right]_0^T =$$

$$= \frac{1}{z} - \lim_{T\to\infty}\frac{e^{-zT}}{z} \Rightarrow \frac{1}{z} \qquad (eq.\ 1)$$

Example 2: $f(t)=e^{at}$ ← a - any number, real or complex

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-zt}e^{at}dt = \int e^{-(z-a)t}dt$$

⇒ this is like $f(t)=1$, when $z \to z-a$

Hence $\underbrace{\mathcal{L}(e^{at}) = \frac{1}{z-a}}_{\text{from eq. (1)}}\left[\operatorname{Re}(z-a)>0;\ \text{i.e., } \operatorname{Re} z > \operatorname{Re} a\right].$

Theory:

eq. 2

Consider $\boxed{f(s) = \int_0^\infty dt\, e^{-st}F(t)}$ ← Laplace transform of $F(t)$

where $F(t)$ diverges exponentially for large t:

$$F(t) = e^{\gamma t}G(t),$$

for some γ, such that $G(t)$ is bounded as $t\to\infty$.