Laplace Transform. Theory

Two examples for $\underbrace{\int_0^\infty f(t)\,e^{-zt}\,dt}_{\text{Laplace integral}} = \lim_{T\to\infty} \int_0^T f(t)\,e^{-zt}\,dt$

$\uparrow$ if this exists

Example 1: $f(t)=1$

$$\mathcal{L}(1) = \int_0^\infty e^{-zt}\,dt = \lim_{T\to\infty}\int_0^T e^{-zt}\,dt = \lim_{T\to\infty}\left[-\frac{e^{-zt}}{z}\right]_0^T =$$

$$= \frac{1}{z} - \lim_{T\to\infty}\frac{e^{-zT}}{z} \Rightarrow \frac{1}{z} \qquad \text{(eq. 1)}$$

Example 2: $f(t)=e^{at}$ $\leftarrow$ a - any number, real or complex

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-zt}e^{at}\,dt = \int_0^\infty e^{-(z-a)t}\,dt$$

$\Rightarrow$ this is like $f(t)=1$, when $z \to z-a$

Hence $\mathcal{L}(e^{at}) = \frac{1}{z-a}$ $[\mathrm{Re}(z-a)>0;$ i.e., $\mathrm{Re}\,z > \mathrm{Re}\,a]$.

$\angle$ from eq. (1)

Theory:

Consider $\boxed{f(s) = \int_0^\infty dt\, e^{-st}\,F(t)}$ eq. 2 $\leftarrow$ Laplace transform of $F(t)$

where $F(t)$ diverges exponentially for large t: $F(t) = e^{\gamma t}G(t)$, for some γ, such that $G(t)$ is bounded as $t \to \infty$.

Then, take:

$$G(t) = \int_{-\infty}^{\infty} dv\, \delta(t-v)\, G(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du\, \overbrace{e^{iu(t-v)}}^{\delta(t-v)} G(v)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} du\, e^{iut} \int_{-\infty}^{\infty} dv\, e^{-iuv} G(v)$$

So that

$$F(t) = e^{\gamma t} G(t) = \frac{e^{\gamma t}}{2\pi} \int_{-\infty}^{\infty} du\, e^{iut} \int_{0}^{\infty} dv\, e^{-iuv} \underbrace{e^{-\gamma v} F(v)}$$

$\left\{ \begin{array}{l} F(v) = e^{\gamma v} G(v) \\ \hookrightarrow G(v) = e^{-\gamma v} F(v) \end{array} \right.$

In terms of $s = \gamma + iu \Rightarrow iu = s - \gamma \Rightarrow$ take e^{st} inside the integral

$$\Rightarrow F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du\, e^{st} \int_{0}^{\infty} dv\, e^{-sv} F(v)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} du\, e^{st} \underbrace{f(s)}_{\text{p.56, eq.2}}$$

However, $s = \gamma + iu$, where γ - some real number - property of the function F.
Integration variable is u: $ds = i\,du$

$$\Rightarrow \boxed{F(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} ds\, e^{st} f(s)}$$

{ Inverse Laplace transform (or Fourier-Mellin integral) or Mellin inverse formula

Clearly, F should not depend on choice of γ. However,
$Re(s) \geq \gamma \Rightarrow$ so that $f(s)$ is defined

Inverting Laplace Integral

$$f(s) = \int_0^\infty dt\, \underbrace{F(t)}_{\text{need to find}}\, e^{-st}$$

$F(t)$ may diverge exponentially, e.g., $F(t) \sim e^{\alpha t}$ for large t.

Then pick at $\beta > \alpha$, such that

$$G(t) \equiv e^{-\beta t} F(t) \to 0 \quad (\text{as } t \to \infty)$$

Also assume $G(t) = 0$ for $t < 0$.

$$\Rightarrow \tilde{G}(t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, e^{iut} \int_{-\infty}^{\infty} dv\, G(v)\, e^{-iuv} =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, e^{iut} \int_0^{\infty} dv\, \underbrace{G(v)}_{e^{-\beta v}F(v)}\, e^{-iuv} = G(t)$$

$$\Rightarrow F(t) = e^{\beta t}\frac{1}{2\pi}\int_{-\infty}^{\infty} du\, e^{iut} \int_0^{\infty} dv\, e^{-\beta v} F(v)\, e^{-iuv}$$

$$= \frac{e^{\beta t}}{2\pi}\int_{-\infty}^{\infty} du\, e^{iut} \underbrace{\int_0^{\infty} dv\, e^{-(\beta+iu)v} F(v)}_{=}$$

$$= \frac{e^{\beta t}}{2\pi}\int_{-\infty}^{\infty} du\, e^{iut} f(\beta + iu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, e^{(\beta+iu)t} f(\beta+iu)$$

Change variables to $s = \beta + iu$, $ds = i\,du$

$$\Rightarrow F(t) = \frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} e^{st} f(s)\, ds$$ – inverse Laplace integral

By definition, β has to be chosen such that all singularities are to the left of $(\mathrm{Re}\, s) = \beta$.

Laplace Method in General

To estimate integrals of the form
$$\int_a^b e^{Mf(x)}dx \quad \text{where } M \text{ - large number}$$

Idea of Laplace:

If $f(x)$ has unique maximum at x_0
$\Rightarrow$ if multiply $Mf(x) \to$ shape does not change as $Mf(x)/Mf(x_0) = f(x)/f(x_0)$,
but $e^{Mf(x)}$ grows exponentially

Theory: Taylor expansion of $f(x)$ near x_0
$$\Rightarrow f'(x_0) = 0 \quad (\max!)$$
$$\Rightarrow \int_a^b e^{Mf(x)}dx \approx e^{Mf(x_0)} \int_a^b e^{\underbrace{-M|f''(x_0)|}_{\text{at } x_0,\ f''<0!}(x-x_0)^2/2}dx$$

Gaussian integral!
$$\Rightarrow \int_a^b e^{Mf(x)}dx \approx \sqrt{\frac{2\pi}{M|f''(x_0)|}}\, e^{Mf(x_0)} \quad \text{as } M \to \infty$$

extension of Laplace method to complex plane
$\Rightarrow$ steepest descent

on complex plane: $\int_a^b e^{Mf(z)}dz \approx \sqrt{\frac{2\pi}{-Mf''(z_0)}}\, e^{Mf(z_0)}$ as $M \to \infty$

Most general: $\oint_C f(z)e^{\lambda g(z)}dz$ where λ - large

Saddle Point Approximation

What is the asymptotic form of $f(t)$ when $t \to \infty$?

↳ otherwise: numerical method

$$\underbrace{\frac{\Omega}{\delta E}}_{F(t)} 2\pi i = \underbrace{\int_{\beta-i\infty}^{\beta+i\infty} d\omega \; e^{\omega E} Z(\omega)}_{\text{inverse } L\text{-integral } [F(t)]} \Rightarrow \int_{-\infty}^{\infty} d\alpha\, e^{\omega E} Z(\omega)$$

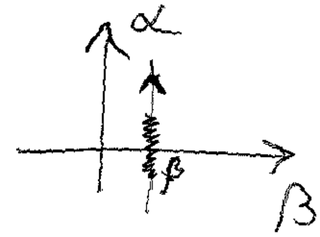
where $\omega = \beta + i\alpha$

$$f(s) = \int_0^\infty dt\, e^{-st} F(t) \quad \text{- Laplace integral}$$

The integral over α involves an oscillatory function since $e^{\omega E} = e^{\beta E} e^{i\alpha E}$

$\Rightarrow$ positive as often as negative

$\Rightarrow$ cancellation except in the neighborhood of $\alpha = 0$



Define:

$$Q = \ln\left[e^{\omega E} Z(\omega)\right] = \omega E + \ln Z(\omega)$$

Approximate Q by expanding around $\alpha = 0$:

$$Q = (\beta + i\alpha)E + \ln Z[f(\beta+i\alpha)] = \overbrace{(\underbrace{\beta}_{\substack{0\text{ order}\\ \beta E}} + i\alpha)E}^{1\text{st order}} + \ln Z(\beta)_{\alpha=0} +$$

$$+ \underbrace{i\alpha}_{1\text{st order}} \left(\frac{\partial \ln Z}{\partial \omega}\right)_{\alpha=0} - \frac{1}{2}\alpha^2 \left(\partial^2 \ln Z / \partial \omega^2\right)_{\alpha=0} + \dots$$

6) However, β is still arbitrary!
So, choose β such that the linear term vanishes — then the approximation will be good:

$$\lim_{\alpha \to 0} [i\alpha E + i\alpha (\partial \ln Z / \partial \omega)_{\alpha=0}]$$

$$\Rightarrow \quad \boxed{E + \left(\frac{\partial \ln Z}{\partial \omega}\right)_{\alpha=0} = 0}$$

For this choice perform the integral

$$e^{\beta E + \ln Z(\beta)_{\alpha=0}} \int_{-\infty}^{\infty} d\alpha \, e^{-\frac{1}{2}\alpha^2 (\partial^2 \ln Z / \partial \omega^2)_{\alpha=0}}$$

$$= e^{\beta E} Z(\beta)_{\alpha=0} \sqrt{2\pi / (\partial^2 \ln Z / \partial \omega^2)_{\beta'=0}}$$

$$\Rightarrow \Omega(E) = e^{\beta E} Z(\beta) \frac{\delta E}{\sqrt{2\pi / B_2}}$$

Leading term

$$\ln \Omega(E) = \beta(E) + \ln Z(\beta)$$

(expression for entropy in canonical ensemble)